

au rang d'une matrice

exercice I

$$1) \text{ soit } \sigma_R(\pi) = \text{Sp}(R^{-1}\pi) \cap \mathbb{C}^*$$

$$\text{On a } \lambda \in \sigma_R(\pi) \Leftrightarrow \lambda \in \mathbb{C}^* \text{ et } \lambda R^{-1}\pi \notin \text{GL}_n(\mathbb{C})$$

$$\Leftrightarrow \lambda \in \mathbb{C}^* \text{ et } R(\lambda I_n - R^{-1}\pi) \notin \text{GL}_n(\mathbb{C})$$

$$\Leftrightarrow \lambda \in \mathbb{C}^* \text{ et } \lambda I_n - R^{-1}\pi \notin \text{GL}_n(\mathbb{C}) \text{ car } R \in \text{GL}_n(\mathbb{C})$$

$$\Leftrightarrow \lambda \in \text{Sp}(R^{-1}\pi) \cap \mathbb{C}^*$$

$$\text{D'autre part } \sigma_R(\pi) = \overline{\text{Sp}(R^{-1}\pi) \cap \mathbb{C}^*}$$

$$b) \text{ si } \pi \text{ est nilpotent } \sigma_R(\pi) = \{0\}$$

Puisque R est inversible $\Leftrightarrow R^{-1}$ est inversible

$$\text{D'autre part } \sigma_R(R^{-1}\pi) = \sigma_R(\pi)$$

$$2) \text{ On a } D = \text{diag}(1, 2, \dots, r, 0, \dots, 0)$$

$$\text{On a } \sigma_R(D) = \lambda = \sigma_R(\pi), \text{ alors } \exists P, Q$$

$$\in \text{GL}_n(\mathbb{C}) \text{ tq } \pi = P D Q.$$

$$b) \text{ On a } \lambda \in \sigma_P Q(\pi) \Leftrightarrow \lambda \in \mathbb{C}^*$$

$$\text{ car } \lambda P Q^{-1} \pi \notin \text{GL}_n(\mathbb{C})$$

$$\text{ On a } \lambda P Q^{-1} \pi = P(\lambda I_n - D)Q \notin \text{GL}_n(\mathbb{C})$$

$$\Leftrightarrow \lambda I_n - D \notin \text{GL}_n(\mathbb{C}) \text{ car } P, Q \in \text{GL}_n(\mathbb{C})$$

$$\Leftrightarrow \text{diag}(\lambda - 1, \lambda - 2, \dots, \lambda - r, \lambda, \lambda, \dots, \lambda)$$

$$\notin \text{GL}_n(\mathbb{C})$$

$$\Leftrightarrow \lambda \in \{1, 2, 3, \dots, r, 0\}$$

$$\text{ car } \lambda \neq 0 \Leftrightarrow \lambda \in \{1, 2, \dots, r\}$$

$$\Leftrightarrow \sigma_{PQ}(\pi) = \{1, 2, 3, \dots, r\}$$

$$\Rightarrow \text{Cand. } (\sigma_{PQ}(\pi) = r)$$

$$c) \text{ On a : } \forall R \in \text{GL}_n(\mathbb{C}), \text{ Cand. } (\sigma_R(\pi)) \leq r$$

$$\Rightarrow r \text{ est un majorant de } \{ \text{Cand. } (\sigma_R(\pi)) : R \in \text{GL}_n(\mathbb{C}) \}$$

$$\text{ De plus d'après, b) } \exists P, Q \in \text{GL}_n(\mathbb{C})$$

$$\text{ tq } \text{Cand. } (\sigma_{PQ}(\pi)) = r.$$

$$\text{ Alors } \max \{ \text{Cand. } (\sigma_R(\pi)) : R \in \text{GL}_n(\mathbb{C}) \} = r$$

$$4) 1) \sigma_R(\pi) = r < n$$

$$a) \text{ L'if n'est inversible.}$$

$$N = (n \times 1) \text{ avec } N = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix} \text{ r fois}$$

$$\text{ On a } \sigma_R(N) = r = \sigma_R(\pi)$$

$$\text{ alors } \exists P, Q \in \text{GL}_n(\mathbb{C}) \text{ tq } \pi = P N Q.$$

$$b) \text{ si } \pi \text{ est nilpotent } \sigma_R(\pi) = \{0\}.$$

$$\text{ On a } \lambda P Q^{-1} \pi \notin \text{GL}_n(\mathbb{C}) \text{ et } \lambda \neq 0$$

$$\Leftrightarrow P(\lambda I_n - N)Q \notin \text{GL}_n(\mathbb{C}) \text{ et } \lambda \neq 0$$

$$\Leftrightarrow \lambda I_n - N \notin \text{GL}_n(\mathbb{C}) \text{ et } \lambda \neq 0$$

$$\Leftrightarrow \text{diag}(\lambda - 1, \lambda - 1, \dots, \lambda - 1, \lambda, \lambda, \dots, \lambda) \notin \text{GL}_n(\mathbb{C}) \text{ et } \lambda \neq 0$$

$$\Leftrightarrow \lambda = 0$$

$$\text{ car } \lambda \neq 0 \text{ et } \lambda \neq 0 \text{ car } \lambda = 0$$

$$\text{ Alors } \sigma_{PQ}(\pi) = \{0\}$$

$$5) \Pi \notin Gl_n(\mathbb{C}) \Leftrightarrow \min\{Cand_{\overline{R}}(\Pi) : R \in Gl_n(\mathbb{C})\} = 0$$

$$\Rightarrow \text{"s\"at } \Pi \notin Gl_n(\mathbb{C}).$$

$$\text{bnc. } Cand_{\overline{R}}(\Pi) \geq 0; \forall R \in Gl_n(\mathbb{C}).$$

$$\text{D'apr\"as 4) } \exists PQ \in Gl_n(\mathbb{C}) \text{ tq } \overline{PQ}(\Pi) = 0$$

$$\text{alors } \min\{Cand_{\overline{R}}(\Pi) : R \in Gl_n(\mathbb{C})\} = 0$$

$$\Leftrightarrow \text{"hyp. } \min\{Cand_{\overline{R}}(\Pi) : R \in Gl_n(\mathbb{C})\} = 0$$

$$\Rightarrow \exists R_0 \in Gl_n(\mathbb{C}) \text{ tq } Cand_{\overline{R_0}}(\Pi) = 0$$

$$\text{Cand}(SP(R_0^{-1}\Pi) \cap \mathbb{C}^*) = 0$$

$$\Leftrightarrow SP(R_0^{-1}\Pi) \cap \mathbb{C}^* = \emptyset$$

$$\Leftrightarrow SP(R_0^{-1}\Pi) = \{0\} \text{ car } SP(R_0^{-1}\Pi) \neq \emptyset$$

$$\Leftrightarrow R_0^{-1}\Pi \text{ est non inversible}$$

$$\Leftrightarrow \Pi \text{ est " " " (} R_0 \text{ inversible)}$$

Partie II

1) hyp f conserve le rang

$$\text{s\"at } \Pi \in Gl_n(\mathbb{C}) \Leftrightarrow \text{rg}(\Pi) = n$$

$$\text{D'o\"u } \text{rg}(f(\Pi)) = n \Leftrightarrow f(\Pi) \text{ inversible}$$

$$\text{D'o\"u } f \text{ conserve l'inversibilit\'e.}$$

$$2) \Pi \text{ non inversible.}$$

$$\lambda R - \Pi \in Gl_n(\mathbb{C}), \text{ alors } f(\lambda R - \Pi) \in Gl_n(\mathbb{C})$$

$$\Rightarrow \lambda f(R) - f(\Pi) \in Gl_n(\mathbb{C}) \text{ (f endo)}$$

$$\Rightarrow \text{bnc } \Pi \text{ est non inversible, alors d'apr. 4)}$$

$$\exists R_1 = PQ \text{ tq } \overline{R_1}(\Pi) = \emptyset$$

$$\Leftrightarrow \forall \lambda \in \mathbb{C}^*: \lambda R_1 - \Pi \in Gl_n(\mathbb{C})$$

$$\text{D'apr\"as 2) } \forall \lambda \in \mathbb{C}^* \lambda f(R_1) - f(\Pi) \in Gl_n(\mathbb{C})$$

$$\text{car } f \text{ conserve l'inversibilit\'e, alors}$$

$$f(R_1) \in Gl_n(\mathbb{C})$$

$$\text{et bnc. } \forall \lambda \in \mathbb{C}^*: \lambda f(R_1) - f(\Pi) \in Gl_n(\mathbb{C})$$

$$\Rightarrow \text{D'o\"u } f(\Pi) = \emptyset \Rightarrow Cand_{\overline{f(\Pi)}}(f(\Pi)) = 0$$

$$\Rightarrow \text{d'apr. I. 5) } \Pi \notin Gl_n(\mathbb{C})$$

$$\Rightarrow \text{D'apr. 2) } \Pi \notin Gl_n(\mathbb{C}) \Rightarrow f(\Pi) \notin Gl_n(\mathbb{C})$$

$$\Leftrightarrow f(\Pi) \in Gl_n(\mathbb{C}) \Rightarrow \Pi \in Gl_n(\mathbb{C})$$

$$\Leftrightarrow f \text{ conserve l'inversibilit\'e, bnc.}$$

$$\text{D'o\"u } f \text{ conserve l'inversibilit\'e.}$$

$$\text{s\"at } \Pi \in \ker f \Leftrightarrow f(\Pi) = 0$$

$$\Leftrightarrow \text{s\"at } \Pi \in \ker f \Leftrightarrow f(\Pi) = 0$$

$$\forall \lambda \in \mathbb{C}^*, \forall R \in Gl_n(\mathbb{C})$$

$$f(\lambda R - \Pi) = \lambda f(R) - f(\Pi) = \lambda f(R)$$

$$\text{Supposons que } \exists R \in Gl_n(\mathbb{C}) : \overline{R}(\Pi) \neq \emptyset$$

$$\Rightarrow \exists \lambda \in \mathbb{C}^*, \text{ tq } \lambda R - \Pi \notin Gl_n(\mathbb{C})$$

$$\text{Or. } f(\lambda R - \Pi) = \lambda f(R) \in Gl_n(\mathbb{C})$$

5) après 3/ $AR - \pi \in GL_n(\mathbb{C})$
 ce qui est impossible; d'où

$$\forall R \in GL_n(\mathbb{C}); \quad \forall R, |\pi| = \emptyset$$

$$\Rightarrow \quad \text{rg}(\pi) = \text{m.v.c.} \quad (\text{car } \text{rg}(\pi) = 0)$$

$$\Leftrightarrow \text{rg}(\pi) = 0 \Rightarrow \pi = 0$$

Ce qui montre que f est injective.

$$5) a) f \in \mathcal{L}(\pi_n(\mathbb{C}))$$

De plus f est injective, et $\dim \pi_n(\mathbb{C}) = n$
 alors f est bijective. $\Rightarrow$ f est un isomorphisme.

morphisme.

$$b) \text{ soit } \pi \in GL_n(\mathbb{C}) \Rightarrow f \circ f^{-1}(\pi) = \pi$$

est inversible, d'où $f^{-1}(\pi)$ est inversible.