

Procédé d'orthogonalisation de Schmidt

Si $(e_1, \dots, e_n)$ est une famille libre de E .

Alors il existe (unique) famille

$(v_1, \dots, v_n)$ orthonormale tq.

$\forall 1 \leq k \leq n; \text{vect}(e_1, \dots, e_k) = \text{vect}(v_1, \dots, v_k)$

avec $v_1 = \frac{e_1}{\|e_1\|}$

$\forall k \leq n; v_k = \frac{e_k - \sum_{i=1}^{k-1} \langle e_k, v_i \rangle v_i}{\|e_k - \sum_{i=1}^{k-1} \langle e_k, v_i \rangle v_i\|}$

$\langle e_k, v_i \rangle = \frac{\langle e_k, v_i \rangle}{\|e_k - \sum_{i=1}^{k-1} \langle e_k, v_i \rangle v_i\|}$

Rq :

$w_i = e_i$
 $\forall 2 \leq k \leq n; w_k = e_k - \sum_{i=1}^{k-1} \frac{\langle e_k, w_i \rangle}{\langle w_i, w_i \rangle} w_i$
 $(w_1, \dots, w_n)$ famille orthogonale

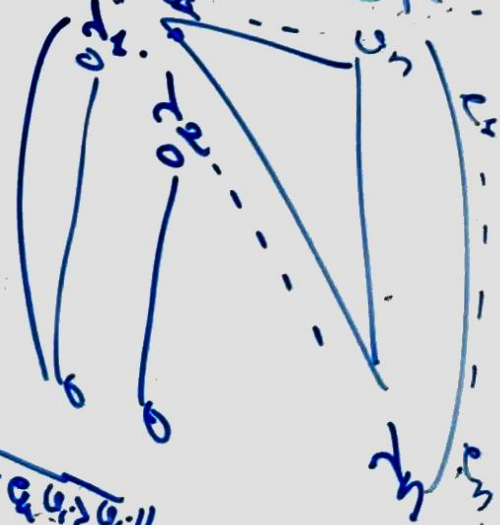
$\rightarrow \text{Suf } F = \text{vect}(e_1, \dots, e_n)$

$F = \text{vect}(v_1, \dots, v_n)$

$B_1 = (e_1, \dots, e_n)$ base de F .

$B_2 = (v_1, \dots, v_n)$ BON de F .

$\Rightarrow P_{B_1} \rightarrow P_{B_2}$



$d_1 = \frac{1}{\|e_1\|} > 0$
 $\forall k \geq 2; d_k = \frac{1}{\|e_k - \sum_{i=1}^{k-1} \langle e_k, v_i \rangle v_i\|}$

$\frac{1}{\|e_k - \sum_{i=1}^{k-1} \langle e_k, v_i \rangle v_i\|}$

Projection orthogonale

Def: Soit F un o.e.v. de E et $\dim F < \infty$

$$\Rightarrow E = F \oplus F^\perp$$

on appelle projection orthogonale sur F .

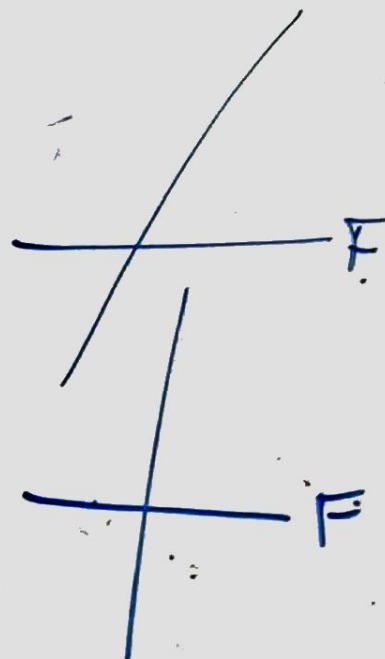
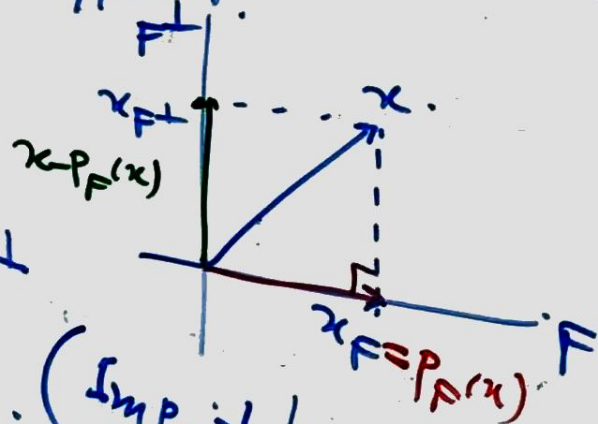
C'est la projection sur F // à $F^\perp$.

$$\forall x \in E; x = x_F + x_{F^\perp}$$

$$P_F(x) = x_F$$

$$\text{Im } P_F = F \text{ et } \ker P_F = F^\perp$$

$$\Rightarrow E = \text{Im } P_F \oplus \ker P_F \quad (\text{Im } P_F \perp \ker P_F)$$



$$\Rightarrow x_F = P_F(x) \quad x_{F^\perp} = x - P_F(x) \quad (x - P_F(x) \in F^\perp)$$

$$\bullet \|x\|^2 = \|P_F(x)\|^2 + \|x - P_F(x)\|^2$$

$$\bullet \langle x - P_F(x), P_F(x) \rangle = 0 \Leftrightarrow \langle x, P_F(x) \rangle = \langle P_F(x), P_F(x) \rangle$$

$$\bullet \forall y \in F; \langle x - P_F(x), y \rangle = 0$$

• soit $B = (e_1, \dots, e_k)$ BON de F .

$$\forall x \in E; \quad P_F(x) = \sum_{i=1}^k \langle x, e_i \rangle e_i$$

Car $P_F(x) \in F \Rightarrow P_F(x) = \sum_{i=1}^k \langle P_F(x), e_i \rangle e_i$

$$\alpha. \quad \langle P_F(x), e_i \rangle = \langle \underbrace{P_F(x) - x}_{\in F^\perp} + x, e_i \rangle_{\substack{\in F \\ \in F}} = \langle x, e_i \rangle.$$

$$\Rightarrow P_F(x) = \sum_{i=1}^k \langle x, e_i \rangle e_i.$$

Alors $\|P_F(x)\|^2 = \sum_{i=1}^k (\langle x, e_i \rangle)^2$

D'après pythagore $\Rightarrow \|P_F(x)\|^2 \leq \|x\|^2$

$$\Rightarrow \sum_{i=1}^k (\langle x, e_i \rangle)^2 \leq \|x\|^2$$

Inégalité de Bessel

→ soit $p \in \mathcal{L}(E)$.

p est une projection orthogonale $\iff$ p est un projecteur ($p \circ p = p$)
et $\text{Im } p \perp \text{Ker } p$

→ $\forall x \in E; \quad d(x, F) = \inf_{y \in F} \|x - y\| = \|x - P_F(x)\|$

$\Rightarrow$ la distance est atteinte en l'unique point $P_F(x)$.

$$\Rightarrow \forall y \in F; \quad \|x - y\| \geq d(x, F) = \|x - P_F(x)\|$$

Symétrie orthogonale

Def: Soit F un o.e.v. de E et $\dim F < \infty$

$$\Rightarrow E = F \oplus F^\perp$$

on appelle symétrie orthogonale par F .

C'est la symétrie $\because F \parallel \alpha F^\perp$.

$$\forall x \in E; x = x_F + x_{F^\perp}$$

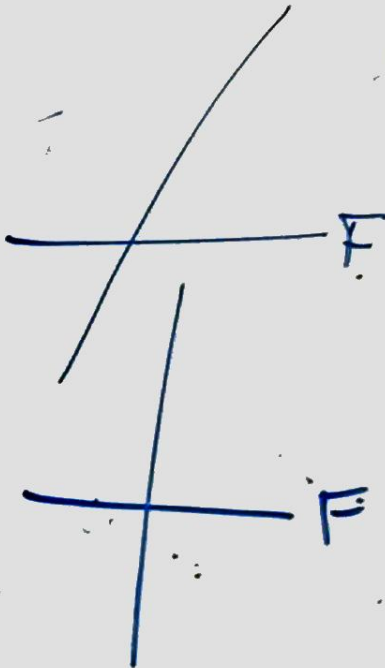
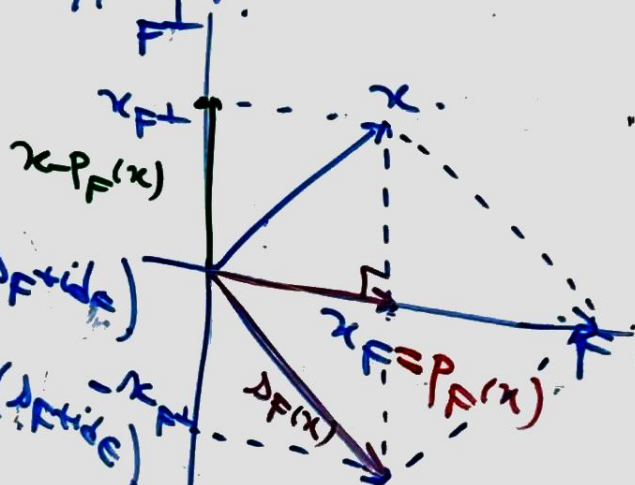
$$s_F(x) = x_F - x_{F^\perp}$$

$$F = \ker(s_F - \text{id}_E) \text{ et } F^\perp = \ker(s_F + \text{id}_E)$$

$$\Rightarrow E = \ker(s_F - \text{id}_E) \oplus \ker(s_F + \text{id}_E)$$

$$\rightarrow \forall x \in E, s_F(x) + x = 2P_F(x)$$

$$s_F = 2P_F - \text{id}_E$$



Exercice 4.

$$P_i(x) = x^i, \quad i \in \{0, 1, 2, 3\}$$

$$1) \quad \alpha_0, \alpha_1, \alpha_2 \in \mathbb{R} \text{ s.t. } \alpha_0 P_0 + \alpha_1 P_1 + \alpha_2 P_2 = 0 \\ \Rightarrow \alpha_0 P_0(x) + \alpha_1 P_1(x) + \alpha_2 P_2(x) = 0, \quad \forall x \in [-1, 1] \\ \alpha_0 + \alpha_1 x + \alpha_2 x^2 = 0, \quad \forall x \in [-1, 1]$$

$$\Rightarrow Q(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 \text{ est } \\ \text{un poly qui admet une infinité de racines} \\ \Rightarrow Q = 0 \Rightarrow \alpha_0 = \alpha_1 = \alpha_2 = 0.$$

(P_0, P_1, P_2) est libre.

$$\text{Or } \langle P_0, P_2 \rangle = \int_0^1 x^2 dx = \frac{1}{3} \neq 0$$

$\Rightarrow (P_0, P_1, P_2)$ n'est pas orthogonale.

$$2) a) \quad \varphi_0 = \frac{P_0}{\|P_0\|} = P_0 = 1.$$

$$\varphi_1 = \frac{P_1 - \langle P_1, \varphi_0 \rangle \varphi_0}{\| \quad \quad \quad \|}$$

$$\varphi_1 = \frac{P_1}{\|P_1\|} = \sqrt{3} \cdot x.$$

$$\varphi_2 = \frac{P_2 - \langle P_2, \varphi_0 \rangle \varphi_0 - \langle P_2, \varphi_1 \rangle \varphi_1}{\| \quad \quad \quad \|}$$

$$\Rightarrow \varphi_2(x) = \frac{3\sqrt{5}}{2} \left(x^2 - \frac{1}{3} \right)$$

b) P_i : proj. orthogonale sur $\mathcal{K}$.

$$P_{\mathcal{K}}(P_2) = \sum_{k=0}^2 \langle P_2, \varphi_k \rangle \varphi_k \\ P_{\mathcal{K}}(P_3) = \sum_{k=0}^3 \langle P_3, \varphi_k \rangle \varphi_k$$

$$d(P_3, F) = \|P_3 - P_F(P_3)\| = \|P_3 - \frac{\sqrt{3}}{5} q_1\|$$

$$= \frac{2}{5\sqrt{5}}$$

$$* \inf \left\{ \int_0^1 (x^3 - ax^2 - bx - c)^2 dx, a, b, c \in \mathbb{R} \right\}$$

$$= \inf \{ \|x^3 - p\|^2, p \in F \}$$

$$= 2 \cdot (d(x^3, F))^2 = \frac{8}{175}$$

Exercice 1 "Série II"

$$E = \{P_n \in \mathcal{S}\}$$

$$\forall P, Q \in E, \langle P, Q \rangle = \int_0^{\infty} P(t)Q(t)e^{-t} dt$$

$$t \mapsto P(t)Q(t)e^{-t} \text{ est } C^\infty \text{ avec}$$

$$\lim_{t \rightarrow \infty} t^p P(t)Q(t)e^{-t} = 0 \Rightarrow$$

$$P(t)Q(t)e^{-t} = o\left(\frac{1}{t^p}\right)$$

$$\text{Or } t \mapsto \frac{1}{t^p} \text{ int\'eg. sur } (0, \infty) \Rightarrow t \mapsto P(t)Q(t)e^{-t}$$

$$\text{est int\'eg. sur } (0, \infty) \text{ d'o\'u}$$

$$\langle \cdot, \cdot \rangle \text{ est bien d\'efini sur } E$$

$$F = \{P \in E; P(0) = 0\}$$

$$(P_0, \dots, P_n) \text{ polynome de 1. d\'egr\'e}$$

$$(1, x, \dots, x^n)$$

$$\Rightarrow \forall 0 \leq k \leq n; \text{vect}(P_0, \dots, P_k) = \text{vect}(1, x, \dots, x^k) = \mathbb{R}_k[x]$$

$$P_0 = 1$$

$$\forall 1 \leq k \leq n; P_k(x) = x^k - \sum_{i=0}^{k-1} \langle x^k, P_i \rangle P_i$$

$$\Rightarrow \deg P_k = k \text{ et } \|P_k\| = 1$$

$$P_{k+1} \in (\mathbb{R}_k[x])^\perp \text{ et } P_{k+1} \perp P_i \text{ } \forall 0 \leq i \leq k$$

2) Req. $(P_k(0))^2 = I$. $\therefore k \in \{-n\}$.

and $\|P_k\|^2 = 1 = \int_0^\infty P_k^e(t) \bar{e}^t dt$.

IPP. $u = P_k^e(t) \rightarrow u' = e P_k(t) P_k'(t)$.

$u' = \bar{e}^t \rightarrow u = -\bar{e}^t$.

$\lim_{t \rightarrow \infty} u(t) u(t) = \lim_{t \rightarrow \infty} -\bar{e}^t P_k^e(t) = 0$

$\Rightarrow \int_0^\infty P_k(t) P_k'(t) \bar{e}^t dt \subset \mathbb{C}$. $1 = [-P_k^e(t) \bar{e}^t]_0^\infty + e \int_0^\infty P_k(t) P_k'(t) \bar{e}^t dt$
 $= P_k^e(0) + e \int_0^\infty P_k(t) P_k'(t) \bar{e}^t dt$

$e \int_0^\infty P_k(t) P_k'(t) \bar{e}^t dt = \langle P_k, P_k' \rangle = 0$

$\Rightarrow (P_k(0))^R = I$

$P_k' \in R_{k+1}[t]$
 $\text{et } P_k \in (P_{k+1}^* \cap \mathbb{N})$

$$3) F = \{P \in \mathcal{P}_n(\mathbb{R}) \mid P(0) = 0\}$$

$$= \text{hyperplan.} = \text{vect}(X, X^2, \dots, X^n)$$

$$\dim F = n \Rightarrow \dim F^\perp = 1$$

$$\Rightarrow F^\perp = \text{vect}(\Phi).$$

$$\Phi \in E \text{ et } (P_0, \dots, P_n) \text{ B.O.N. de } E.$$

$$\Rightarrow \Phi = \sum_{k=0}^n \langle \Phi, P_k \rangle \cdot P_k.$$

$$\text{or } P_k(X) - P_k(0) \in F, \quad 0 \leq k \leq n.$$

$$\Rightarrow \langle \Phi, P_k(X) - P_k(0) \rangle = 0$$

$$\quad \quad \quad \leftarrow \Phi \perp F$$

$$\langle \Phi, P_k \rangle = \langle \Phi, P_k(0) \rangle = P_k(0) \langle \Phi, 1 \rangle$$

$$\Rightarrow \Phi = \sum_{k=0}^n P_k(0) \cdot \langle \Phi, 1 \rangle \cdot P_k$$

$$\Phi = \langle \Phi, 1 \rangle \cdot \sum_{k=0}^n P_k(0) P_k$$

$$* d(1, F) = \|1 - P_F(1)\|$$

avec P_F : orthogonal sur F

$$1 = \underbrace{1 - P_F(1)}_{1_{F^\perp}} + P_F(1)$$

$$1_{F^\perp} + 1_F.$$

$$d(1, F) = \|1_{F^\perp}\|.$$

$$2 \left(\frac{\Phi}{\|\Phi\|} \right) \text{ B.O.N. de } F^\perp.$$

$$\Rightarrow 1_{F^\perp} = \left\langle 1, \frac{\Phi}{\|\Phi\|} \right\rangle \cdot \frac{\Phi}{\|\Phi\|}$$

$$= \frac{\langle 1, \Phi \rangle \cdot \Phi}{\|\Phi\|^2}$$

$$2 \|\Phi\|^2 = \langle \Phi, 1 \rangle^2 \sum_{k=0}^n \frac{(P_k(0))^2}{(n+1)}$$

$$= \langle \Phi, 1 \rangle^2 \frac{1}{(n+1)} \sum_{k=0}^n (P_k(0))^2$$

$$\left(\frac{\|\Phi\|}{\sqrt{n+1}} \right)^2$$

$$\Rightarrow \mathbf{1}_{F^\perp} = \frac{Q}{\langle Q, \mathbf{1} \rangle (n+1)} \Rightarrow \|\mathbf{1}_{F^\perp}\| = \frac{\|Q\|}{|\langle Q, \mathbf{1} \rangle| (n+1)}$$

$$d(\mathbf{1}, F) = \frac{1}{\sqrt{n+1}}$$

$$\cdot d(\mathbf{1}, F^\perp) = \|\mathbf{1}_F\| =$$

$$\text{or } \|\mathbf{1}\|^2 = \|\mathbf{1}_F\|^2 + \|\mathbf{1}_{F^\perp}\|^2$$

$$\int_0^\infty e^{-t} dt = 1 \Rightarrow \|\mathbf{1}_F\| = \sqrt{1 - \frac{1}{n+1}} = \sqrt{\frac{n}{n+1}}$$

Exercise 1: Positive Definite Gram

1) $u, v \in E$

$$G(u, v) = \begin{pmatrix} \langle u, u \rangle & \langle u, v \rangle \\ \langle v, u \rangle & \langle v, v \rangle \end{pmatrix}$$

$$= \langle u, u \rangle \langle v, v \rangle - \langle u, v \rangle^2$$

$$= \|u\|^2 \|v\|^2 - (\langle u, v \rangle)^2$$

Cauchy-Schwarz inequality.

$$(\langle u, v \rangle)^2 \leq \|u\|^2 \|v\|^2$$

$$\Rightarrow G(u, v) \geq 0$$

$$G(u, v) = 0 \Leftrightarrow u \text{ et } v \text{ sont linéaires}$$

2) $G(u, v, w) = \begin{pmatrix} \langle u, u \rangle & \langle u, v \rangle & \langle u, w \rangle \\ \langle v, u \rangle & \langle v, v \rangle & \langle v, w \rangle \\ \langle w, u \rangle & \langle w, v \rangle & \langle w, w \rangle \end{pmatrix}$

3) $\langle w, v \rangle = 0 \text{ et } \langle w, u \rangle = 0$

$$G(u, v, w) = \begin{pmatrix} \langle u, u \rangle & \langle u, v \rangle & 0 \\ \langle u, v \rangle & \langle v, v \rangle & 0 \\ 0 & 0 & \langle w, w \rangle \end{pmatrix}$$

$$= \|w\|^2 \cdot G(u, v)$$

b) $w \in \text{vect}(u, v)$

$$w = \alpha u + \beta v$$

$$\Rightarrow C_3 = \alpha C_1 + \beta C_2$$

$$\Rightarrow \det(C_1, C_2, C_3) = 0$$

$$\Rightarrow G(u, v, w) = 0$$

3)

$$G(u_1, \dots, u_n) = \begin{pmatrix} \langle u_1, u_1 \rangle & \dots & \langle u_1, u_n \rangle \\ \vdots & \ddots & \vdots \\ \langle u_n, u_1 \rangle & \dots & \langle u_n, u_n \rangle \end{pmatrix}$$

Si $(u_1, \dots, u_n)$ est liée.
 $\Rightarrow \exists i_0 \in \{1, \dots, n\}$ tq. $u_{i_0} = \sum_{\substack{i=1 \\ i \neq i_0}}^n d_i u_i$

$$\Rightarrow L_{i_0} = \sum_{\substack{i=1 \\ i \neq i_0}}^n d_i L_i$$

Donc $G(u_1, \dots, u_n) = 0$

4) Si $(u_1, \dots, u_n)$ est libre, on pose $F = \text{vect}(u_1, \dots, u_n)$ ($\dim F = n$)
 $B_1 = (e_1, \dots, e_n)$ base de F . $A = P_{B_2} \rightarrow B_1$

$$A = \begin{pmatrix} u_1 \cdots u_n \\ \vdots \\ \langle u_j, e_i \rangle \end{pmatrix} \begin{array}{c} e_1 \\ \vdots \\ e_i \\ \vdots \\ e_n \end{array}$$

$\forall i, j \leq n$
 $u_j = \sum_{i=1}^n \langle u_j, e_i \rangle e_i$

on a $(\text{Gram}(U_1, \dots, U_n))_{ij} = \langle U_i, U_j \rangle$.

D'autre part on a :

$$\begin{aligned} (A \cdot A)_{ij} &= \sum_{k=1}^n (A)_{ik} (A)_{kj} \\ &= \sum_{k=1}^n \langle U_i, e_k \rangle \langle U_j, e_k \rangle \end{aligned}$$

(BÖV) $\langle U_i, U_j \rangle = \langle U_i, U_j \rangle$.

$$\begin{aligned} \langle x, y \rangle &= \sum x_i y_i \\ &= \sum_{i=1}^n \langle x, e_i \rangle \langle y, e_i \rangle \end{aligned}$$

D'où :

i) on a

$$(U_1, \dots, U_n) = (A \cdot A)$$

$$\begin{aligned} G(U_1, \dots, U_n) &= \det(\text{Gram}(U_1, \dots, U_n)) \\ &= \det(A \cdot A) = (\det A)^2 \end{aligned}$$

or $A = P_{B_1 \rightarrow B_2} \Rightarrow \text{universale} \Rightarrow \det A \neq 0$

D'où $G(U_1, \dots, U_n) > 0$

g) $x \in E, m$.

$$d(x, F)^2 = \frac{G(x, U_1, \dots, U_n)}{G(U_1, \dots, U_n)}$$

on a $G(x, U_1, \dots, U_n) = \left(\sum_{i=1}^n \langle x, U_i \rangle \langle U_i, U_j \rangle \right)^2 \cdot G(U_1, \dots, U_n)$

$$\begin{pmatrix} \langle x, x \rangle & \langle x, U_1 \rangle & \dots & \langle x, U_n \rangle \\ \langle U_1, x \rangle & \langle U_1, U_1 \rangle & \dots & \langle U_1, U_n \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle U_n, x \rangle & \langle U_n, U_1 \rangle & \dots & \langle U_n, U_n \rangle \end{pmatrix}$$

Satz P_F : orthogonalproj. auf F .

$$\rightarrow d(x, F) = \|x - P_F(x)\|^2.$$

$$\|x\|^2 = \|x - P_F(x)\|^2 + \|P_F(x)\|^2$$

$$\forall 1 \leq k \leq n; \quad \langle U_k, x \rangle = \langle U_k, P_F(x) \rangle. \quad (x - P_F(x) \perp U_k)$$

Dann:

$$Gram(x, U_1, \dots, U_n) = \begin{pmatrix} \|x - P_F(x)\|^2 + \|P_F(x)\|^2 & \langle P_F(x), U_1 \rangle & \dots & \langle P_F(x), U_n \rangle \\ \langle U_1, P_F(x) \rangle & & & \\ \vdots & & & \\ \langle U_n, P_F(x) \rangle & & & \end{pmatrix}$$

$$\Rightarrow \det Gram(x, U_1, \dots, U_n)$$

$$Gram(U_1, \dots, U_n).$$

$$\det \begin{pmatrix} \|x - P_F(x)\|^2 & \langle U_1, P_F(x) \rangle \\ \vdots & \vdots \\ \langle U_n, P_F(x) \rangle & \vdots \end{pmatrix} \times \det \begin{pmatrix} \langle P_F(x), P_F(x) \rangle & \langle P_F(x), U_1 \rangle & \dots & \langle P_F(x), U_n \rangle \\ \langle U_1, P_F(x) \rangle & & & \\ \vdots & & & \\ \langle U_n, P_F(x) \rangle & & & \end{pmatrix}$$

$$= \|x - p_F(x)\|^{\alpha} G(u_1 - ; u_n) \\ + \underbrace{G(p_F(x), u_1 - u_n)}_{\ll 0}$$

Can $p_F(x) \in \text{vert}(u_1 - u_n)$