

Rappel: "Système différentiel"

$$(S): \begin{cases} x_1'(t) = a_{11}(t)x_1(t) + \dots + a_{1n}(t)x_n(t) + b_1(t) \\ \vdots \\ x_n'(t) = a_{n1}(t)x_1(t) + \dots + a_{nn}(t)x_n(t) + b_n(t) \end{cases}$$

$$(S) \Leftrightarrow X'(t) = A(t)X(t) + B(t)$$

avec $X(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}$; $A(t) = (a_{ij}(t))_{i,j}$

$\rightarrow (H): X'(t) = A(t)X(t)$ $B(t) = \begin{pmatrix} b_1(t) \\ \vdots \\ b_n(t) \end{pmatrix}$

$(E_n): y^{(n)}(t) + a_{n-1}(t)y^{(n-1)}(t) + \dots + a_0(t)y(t) = b(t)$

Soit $X(t) = \begin{pmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{pmatrix}$

$$X'(t) = \begin{pmatrix} y_1'(t) \\ \vdots \\ y_n'(t) \end{pmatrix} = \begin{pmatrix} a_{11}(t) & \dots & a_{1n}(t) \\ \vdots & \ddots & \vdots \\ -a_{n1}(t) & \dots & -a_{nn}(t) \end{pmatrix} \begin{pmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{pmatrix} + \begin{pmatrix} b_1(t) \\ \vdots \\ b_n(t) \end{pmatrix}$$

$$X'(t) = A(t) \cdot X(t) + B(t)$$

- Th.
- ① L'ensemble des solutions de (H) est un K. e. v.
- ② L'ensemble des solutions de (S) est $\{ X(t) = X_H(t) + X_P(t) \}$ avec X_H solution de (H) et X_P solution particulière de (S).

Th de Cauchy-Lipschitz

$$\begin{cases} X'(t) = A(t)X(t) & (H) \\ X(t_0) = X_0. \end{cases}$$

admet une unique solution

$$\Rightarrow \Phi_{t_0} : S_H \longrightarrow \Pi_n(K)$$
$$X \mapsto X(t_0)$$

est un isomorphisme

$$\rightarrow \boxed{\dim S_H = n}$$

on déduit

$$(H_n) : y^{(n)}(t) + a_{n-1}(t)y^{(n-1)}(t) + \dots + a_0(t)y(t) = 0$$

$$\dim(S_{H_n}) = n$$

Real

$(x_1, x_2, \dots, x_n)$ un système fondamental de solutions de (H)

$\Leftrightarrow (x_1, x_2, \dots, x_n)$ base de S_H .

$\Leftrightarrow \det(x_1(t), \dots, x_n(t)) \neq 0$
" Wronskien"

$W(t) = \text{Wronskien}$.

Th. $(x_1, x_2, \dots, x_n)$ base de S_H .

$\Leftrightarrow W(t) \neq 0, \forall t \in I$

$\Leftrightarrow \exists t_0 \in I$ tel que $W(t_0) \neq 0$

Prop. $y''(t) + a(t)y'(t) + b(t)y(t) = 0$ (H₂).

$\Leftrightarrow X'(t) = \begin{pmatrix} 0 & 1 \\ -b(t) & -a(t) \end{pmatrix} X(t)$ avec $X(t) = \begin{pmatrix} y(t) \\ y'(t) \end{pmatrix}$

(y_1, y_2) système fondamental de solutions de (H₂)

$\Leftrightarrow W(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} \neq 0$

Th. (y_1, y_2) base de S_{H_2}

$\Leftrightarrow W(t) \neq 0, \forall t \in I$

$\Leftrightarrow \exists t_0 \in I$ tel que $W(t_0) \neq 0$

Enf. et

W est dérivable sur I
 $\forall t \in I, W'(t) =$

$\begin{vmatrix} y_1'(t) & y_2'(t) \\ y_1''(t) & y_2''(t) \end{vmatrix} = \begin{vmatrix} y_1'(t) & y_2'(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} + \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = 0 + W(t) = W(t)$
(det est brisé!)

$$= -y_2(t) \cdot (-a(t)y_1'(t) - b(t)y_1(t)) \\ + y_1(t) \cdot (-a(t)y_2'(t) - b(t)y_2(t))$$

$$\Rightarrow a(t) (y_2(t)y_1'(t) - y_1(t)y_2'(t)).$$

$$w'(t) = -a(t)w(t).$$

$$\Rightarrow w(t) = \lambda e^{-\int a(t) dt}.$$

$$w(t_0) = 0 \Leftrightarrow \lambda = 0 \Leftrightarrow \forall t \in I, w(t) = 0$$

App (H): $ay'' + by' + cy = 0$
 (E): $ar^2 + br + c = 0$

• $\Delta > 0 \rightarrow r_1 \neq r_2$

$$y_1(x) = e^{r_1 x}$$

$$y_2(x) = e^{r_2 x}$$

$S_H = \{ y_h(x) = \alpha e^{r_1 x} + \beta e^{r_2 x}, \alpha, \beta \in \mathbb{R} \}$

$$w(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{r_1 x} & e^{r_2 x} \\ r_1 e^{r_1 x} & r_2 e^{r_2 x} \end{vmatrix} = (r_2 - r_1) e^{(r_1 + r_2)x} \neq 0$$

$$\rightarrow y''(t) + a(t)y'(t) + b(t)y(t) = c(t) \quad (E)$$

(y_1, y_2) base of S_H .

$$y_h(t) = \alpha y_1(t) + \beta y_2(t).$$

on cherche $y_p(t) = \alpha(t)y_1(t) + \beta(t)y_2(t)$.

solution de (E):

$$\alpha'(t)x_1(t) + \beta'(t)x_2(t) = B(t).$$

$$\alpha'(t) \begin{pmatrix} y_1(t) \\ y_1'(t) \end{pmatrix} + \beta'(t) \begin{pmatrix} y_2(t) \\ y_2'(t) \end{pmatrix} = \begin{pmatrix} 0 \\ c(t) \end{pmatrix}$$

$$\rightarrow \begin{cases} \alpha'(t)y_1(t) + \beta'(t)y_2(t) = 0 \\ \alpha'(t)y_1'(t) + \beta'(t)y_2'(t) = c(t) \end{cases}$$

$$\alpha'(t) = \frac{\begin{vmatrix} 0 & y_2(t) \\ c(t) & y_2'(t) \end{vmatrix}}{W(t)} \quad \beta'(t) = \frac{\begin{vmatrix} y_1(t) & 0 \\ y_1'(t) & c(t) \end{vmatrix}}{W(t)}$$

Solution particulière

Soit $(x_1, \dots, x_n)$ base de S_H .

$$x_H(t) = \sum_{k=1}^n \alpha_k x_k(t).$$

On cherche $x_p(t) = \sum_{k=1}^n \alpha_k(t) x_k(t)$.

solution de (S) $\Leftrightarrow$

$$x_p'(t) = A(t) x_p(t) + B(t)$$

$$\Leftrightarrow \sum_{k=1}^n (\alpha_k'(t) x_k(t) + \alpha_k(t) x_k'(t)) = A(t) \sum_{k=1}^n \alpha_k(t) x_k(t) + B(t)$$

(L'app. $K \times \Pi_n(K) \rightarrow \Pi_n(K)$
 $(d, v) \mapsto d.v$ bilinéaire)

$$\sum_{k=1}^n \alpha_k'(t) x_k(t) + \sum_{k=1}^n \alpha_k(t) x_k'(t) = \sum_{k=1}^n \alpha_k(t) A(t) x_k(t) + B(t)$$

$$\sum_{k=1}^n \alpha_k'(t) x_k(t) = B(t)$$

$$\alpha_k'(t) = \frac{\det(x_1(t), \dots, \overset{k}{x_k(t)}, \dots, x_n(t))}{w(t)}$$

$$\left\{ \begin{array}{l} \alpha_1'(t) x_1(t) + \dots = b_1(t) \\ \alpha_2'(t) x_2(t) + \dots \\ \vdots \\ \alpha_n'(t) x_n(t) + \dots = b_n(t) \end{array} \right.$$

Exercice 2 "serie I"

(E): $x^2 y'' + 4xy' + 2y = \ln(1+x)$

1) et 2) $y_p(x) =$

3) (H): $x^2 y'' + 4xy' + 2y = 0$

$y(x) = x^n$ solution de (H)

$\Leftrightarrow (n(n-1) + 4n + 2) x^n = 0$

$\Leftrightarrow x \neq 0$
 $n^2 + 3n + 2 = 0$

$$r = -1 \text{ ou } r = -2$$

$\Rightarrow y_1(x) = \frac{1}{x}$ et $y_2(x) = \frac{1}{x^2}$ sont des solutions de (1).

et on a

$$W(y_1(x), y_2(x)) = \begin{vmatrix} \frac{1}{x} & \frac{1}{x^2} \\ -\frac{1}{x^2} & -\frac{2}{x^3} \end{vmatrix} = -\frac{1}{x^4} \neq 0$$

$\Rightarrow (y_1, y_2)$ est libre.
et on a $\dim S_H = 2 \Rightarrow (y_1, y_2)$ base de S_H .

Ainsi, $S_E = \left\{ y(x) = \frac{\alpha}{x} + \frac{\beta}{x^2} + y_p(x), \alpha, \beta \in \mathbb{R} \right\}$

Systèmes diff à coefficients constants

(H): $X'(t) = A X(t)$

Thm 1: $S_H = \{ X(t) = e^{tA} \cdot u, u \in \mathbb{R}^n(K) \}$

Th 2 Si $(v_1, \dots, v_n)$ base de $\mathbb{R}^n(K)$
Alors $(e^{tA} \cdot v_1, \dots, e^{tA} \cdot v_n)$ base de S_H .

Effet: $\forall k \in \{1, \dots, n\}, e^{tA} \cdot v_k \in S_H$.

$\alpha_1, \dots, \alpha_n \in K$ tq. $\sum_{i=1}^n \alpha_i e^{tA} \cdot v_i = 0$

$e^{tA} \cdot (\sum_{i=1}^n \alpha_i v_i) = 0 \Rightarrow \sum_{i=1}^n \alpha_i v_i = 0$ car $(e^{tA})_{t=0} = I$
substit $t=0$
 $\sum_{i=1}^n \alpha_i v_i = 0$ car $(e^{tA})_{t=0} = I$

substit
 $\omega(t) = \det(e^{tA} \cdot v_1, \dots, e^{tA} \cdot v_n)$
 $\omega(0) = \det(v_1, \dots, v_n) \neq 0$

ad $S_H = n \Rightarrow (\dots, v_k - e^{tA} v_k) \in S_H$

Th 3 Si A est diagonalisable
 $\Rightarrow \exists (v_1, \dots, v_n)$ base de vecteurs propres de A .

Th 2 $\Rightarrow (e^{tA} \cdot v_1, \dots, e^{tA} \cdot v_n)$ base de S_H .
ad $e^{tA} \cdot v_k = e^{\lambda_k t} \cdot v_k$.
Donc:

$S_H = \{ X(t) = \sum_{k=1}^n \alpha_k e^{\lambda_k t} \cdot v_k, \alpha_k \in K \}$

$\Rightarrow$ Si A est trigonalisable:
 $\Rightarrow \exists P \in GL_n(K), \exists T$ triangulaire
 $A = P T P^{-1}$

$$X'(t) = A X(t) = P T P^{-1} X(t)$$

$$\Leftrightarrow P^{-1} X'(t) = T P^{-1} X(t)$$

$$\Leftrightarrow Y'(t) = T Y(t) \text{ avec } Y(t) = P^{-1} X(t)$$

$$\Leftrightarrow \begin{pmatrix} y_1'(t) \\ \vdots \\ y_n'(t) \end{pmatrix} = \begin{pmatrix} \triangle \\ \bigcirc \end{pmatrix} \begin{pmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{pmatrix}$$

Remarque: $n \in \mathbb{N}^*$.

Soit $A \in M_n(\mathbb{K})$ q. $\chi_A(x) = (x-d)^n$ et $A \neq dI_n$.
Résoudre: $X'(t) = A X(t)$.

$$\text{On a } \chi_A(A) = 0_n \Rightarrow (A - dI_n)^n = 0_n$$

$$\Rightarrow e^{A-dI_n} = \sum_{k=0}^{n-1} \frac{(A-dI_n)^k}{k!}$$

$$\begin{aligned} \Rightarrow e^A &= e^{A-dI_n + dI_n} \\ &= e^{dI_n} e^{(A-dI_n)} \quad (\text{Commute}) \\ &= e^d \cdot e^{(A-dI_n)} \\ &= e^d \cdot \sum_{k=0}^{n-1} \frac{(A-dI_n)^k}{k!} \end{aligned}$$

Exercice 5

$$A = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & -1 \\ -1 & 1 & 3 \end{pmatrix}$$

1) $\chi_A(x) = (x-2)^2(x-1)$

$E_2 = \text{vect}(v_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}) \Rightarrow \dim E_2 = 1 \neq \text{mult}_2$

$\Rightarrow A$ n'est pas diagonalisable.

2) Rq. $\exists P \in GL_3(\mathbb{R})$ tq.

$$A = P \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} P^{-1}$$

soit $P \in GL_3(\mathbb{R})$ tq $\text{Mat}(P, B_C) = A$

$\exists ? B = (v_1, v_2, v_3)$ tq $\text{Mat}(P, B) = T$.

$P(v_1) = v_1 \Rightarrow v_1 \in E_1 = \text{vect}\left(\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}\right)$
 $P(v_2) = v_2 \Rightarrow v_2 \in E_1$
 $P(v_3) = v_2 + v_3$

soit $v_3 = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ tq $P(v_3) = v_2 + 2v_3$

$$\Leftrightarrow A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + 2 \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\Leftrightarrow \begin{cases} y + 2z = 1 + 2x \\ x + 2y - z = -2y \\ -x + y + 3z = 1 + 2z \end{cases}$$

$$\Leftrightarrow \begin{cases} y = 1 \\ x = 3 \\ \text{arbitraire} \end{cases}$$

on prend $v_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \Rightarrow (v_1, v_2, v_3)$ base

$$\exists P = P_{B_C \rightarrow B} = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 2 & 0 \end{pmatrix}$$

$A = \text{Mat}(P, B_C) = P_{B_C \rightarrow B} \text{Mat}(P, B) P_{B \rightarrow B_C}$
 $A = P T P^{-1}$

3) 5) ⇔

$$X'(t) = A X(t) = P T P^{-1} X(t)$$

$$\Leftrightarrow P^{-1} X'(t) = T P^{-1} X(t)$$

$$\Leftrightarrow Y'(t) = T Y(t) \text{ avec } Y(t) = P^{-1} X(t)$$

$$Y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{pmatrix}$$

$$\Leftrightarrow y_1(t) = d_1 e^t$$

$$\Leftrightarrow \begin{cases} y_1'(t) = y_1(t) \\ y_2'(t) = 2y_2(t) + y_3(t) \\ y_3'(t) = 2y_3(t) \end{cases}$$

$$\Leftrightarrow y_3(t) = d_3 e^{2t}$$

$$\textcircled{1} \begin{cases} y_2'(t) = 2y_2(t) + d_3 e^{2t} \\ y_{2,h}(t) = d_2 e^{2t} \end{cases}$$

$$y_{2,h}(t) = d_2 e^{2t}$$

on cherche $y_{2,p}(t) = d_2(t) e^{2t}$ avec

$$d_2'(t) = \frac{d_2(t) e^{2t}}{e^{2t}} = d_2$$

$$\Rightarrow d_2(t) = d_3 t + c t^2$$

$$\Rightarrow y_{2,p}(t) = d_3 t e^{2t}$$

$$\Rightarrow y_2(t) = d_2 e^{2t} + d_3 t e^{2t}$$

$$\Rightarrow X(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} = \begin{pmatrix} -3 & 1 & 0 \\ 1 & 0 & 1 \\ -2 & 1 & 0 \end{pmatrix} \begin{pmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{pmatrix}$$