

Correction

1. $x \mapsto \frac{1}{(chx)^n}$ est continue sur $[0, +\infty[$.

au v(+∞) : $chx \underset{+\infty}{\sim} e^x \Rightarrow (chx)^n \underset{+\infty}{\sim} e^{nx}$

D'où $\frac{1}{(chx)^n} \underset{+\infty}{\sim} e^{-nx}$.

Où $x^2 e^{-nx} \xrightarrow{x \rightarrow \infty} 0, \forall n \in \mathbb{N}^* \Rightarrow e^{-nx} \underset{+\infty}{\sim} o\left(\frac{1}{x^2}\right)$

$\Rightarrow \int_1^{+\infty} e^{-nx} dx$ cv, puisque $\int_1^{+\infty} \frac{1}{x^2} dx$ cv.

Ainsi $\int_1^{+\infty} \frac{1}{(chx)^n} dx$ cv $\Rightarrow \int_0^{+\infty} \frac{1}{(chx)^n} dx$ cv, $\forall n \in \mathbb{N}^*$.

Ce qui montre que F_n existe, $\forall n \in \mathbb{N}^*$.

2. a) $F_1 = \int_0^{+\infty} \frac{1}{chx} dx = \int_0^{+\infty} \frac{2}{e^x + e^{-x}} dx$

On pose $t = e^x \Rightarrow dt = e^x dx = t dx$.

$x \rightarrow 0 \Rightarrow t \rightarrow 1$

$x \rightarrow +\infty \Rightarrow t \rightarrow +\infty$

D'où $F_1 = \int_1^{+\infty} \frac{2}{t + \frac{1}{t}} \cdot \frac{1}{t} dt = \lim_{x \rightarrow +\infty} 2 [\arctg t]_1^x = 2\left(\frac{\pi}{2} - \frac{\pi}{4}\right) =$

$\frac{\pi}{2} \Rightarrow F_1 = \frac{\pi}{2}$

b)

$$\begin{aligned} F_2 &= \int_0^{+\infty} \frac{1}{(chx)^2} dx \\ &= \int_0^{+\infty} \frac{4}{(e^x + e^{-x})^2} dx \\ &= \int_0^{+\infty} \frac{4}{(e^{2x} + 2 + e^{-2x})} dx \end{aligned}$$

$t = e^{2x} \Rightarrow dt = 2e^{2x} dx = 2t dx$

$$\begin{aligned} F_2 &= \int_1^{+\infty} \frac{4}{t + 2 + \frac{1}{t}} \cdot \frac{dt}{2t} \\ &= \int_1^{+\infty} \frac{2dt}{t^2 + 2t + 1} \\ &= 2 \int_1^{+\infty} \frac{1}{(t+1)^2} \\ &= 2 \lim_{x \rightarrow +\infty} \left[-\frac{1}{t+1} \right]_1^x \\ &= \frac{2}{2} - \frac{2}{2} = 0 \end{aligned}$$

3.

$$\begin{aligned} F_{2n+1} &= \int_0^{+\infty} \frac{dx}{(chx)^{2n+1}} \\ &= \int_0^{+\infty} (chx)^{-n-2} \\ &= \int_0^{+\infty} (chx)^{-n-2} (ch^2x - sh^2x) dx \\ &= \int_0^{+\infty} (chx)^{-n} dx - \int_0^{+\infty} (chx)^{-n-2} sh^2x dx \end{aligned}$$

$$\begin{cases} U = shx & V' = (chx)^{-n-2} shx \\ U' = chx & V = \frac{1}{-n-4} (chx)^{-n-1} \end{cases}$$

$$\begin{aligned} &= F_n + \lim_{t \rightarrow +\infty} \left(\left[\frac{-shx(chx)^{-n-1}}{n+1} \right]_0^t \right) - \frac{1}{n+1} \int_0^t (chx)^{-n} dx \\ &sh(0) = 0, sh t(ch t)^{-n-1} \sim e^t e^{-n} = e^{-nt} \xrightarrow{n \rightarrow +\infty} 0 \end{aligned}$$

$$\text{D'où } F_{n+2} = F_n - \frac{1}{n+1} F_n.$$

$$F_{n+2} = \frac{n}{n+1} F_n.$$

$$4. \text{ chx} > 1 \Rightarrow \frac{1}{\text{chx}} \leq 1. \text{ D'où } \frac{1}{(\text{chx})^{n+1}} \leq \frac{1}{(\text{chx})^n} \Rightarrow F_{n+1} \leq F_n.$$

$$5. \text{ a) } u_n = n F_n F_{n+1}.$$

$$u_1 = F_1 F_2 = \frac{\pi}{2}$$

Soit $n \in \mathbb{N}^*$,

$$u_{n+1} = (n+1) F_{n+1} F_{n+2} = (n+1) F_{n+1} \frac{n}{n+1} F_n = n F_n F_{n+1} = u_n$$

D'où la suite (u_n) est constante. $\forall n \in \mathbb{N}^*$, $u_n = u_1 = \frac{\pi}{2}$.

b) On a $1 \geq \frac{F_{n+1}}{F_n} \geq \frac{F_{n+2}}{F_n}$ car la suite (F_n) est décroissante.

D'où $\frac{n}{n+1} \leq \frac{F_{n+1}}{F_n} \leq 1$. Alors $\lim_{n \rightarrow +\infty} \frac{F_{n+1}}{F_n} = 1$

c) On a $u_n = n F_n F_{n+1} = \frac{\pi}{2}$ et $F_{n+1} \sim F_n$

$$\Rightarrow n F_n^2 \sim \frac{\pi}{2} \Rightarrow F_n \sim \sqrt{\frac{\pi}{2n}}$$

$$6. \bullet F_{2n+2} = \frac{2n}{2n+1} F_{2n}, \prod_{k=1}^{n-1} F_{2k+2} = \prod_{k=1}^{n-1} \frac{2k}{2k+1} F_{2k}$$

$$F_{2n} = \frac{2.4 \dots 2n-2}{3.5 \dots 2n-1} F_2 = \frac{(2^{n-1} (n-1)!)^2}{(2n-1)!}$$

$$\bullet F_{2n+1} = \frac{2n-1}{2n} F_{2n-1}, \prod_{k=1}^n F_{2k+1} = \prod_{k=1}^n \frac{2k-1}{2k} F_{2k-1}$$

$$\Rightarrow F_{2n+1} = \frac{1.3 \dots 2n-1}{2.4 \dots 2n} F_1 = \frac{\pi}{2}$$

$$F_{2n+1} = \frac{(2n)!}{(2^n n!)^2} \frac{\pi}{2}$$

$$7. \text{ a) } v_n = \ln(n^n \sqrt{n} e^{-n}) - \ln(n!)$$

$$v_{n+1} - v_n = \ln\left(\frac{(n+1)^{n+1} \sqrt{n+1} e^{-(n+1)}}{(n+1)!}\right) - \ln\left(\frac{n^n \sqrt{n} e^{-n}}{n!}\right)$$

$$= \ln\left(\frac{(n+1)^{n+1} \sqrt{n+1} e^{-(n+1)}}{(n+1)!} \frac{n!}{n^n \sqrt{n} e^{-n}}\right)$$

$$= \ln\left(\frac{(n+1)^{n+1} \sqrt{n+1} e^{-1}}{n^n (n+1) \sqrt{n}}\right)$$

$$= \ln\left(\left(\frac{n+1}{n}\right)^n \sqrt{\frac{n+1}{n}} e^{-1}\right)$$

$$= \ln\left(\left(\frac{n+1}{n}\right)^{n+\frac{1}{2}}\right) - 1$$

$$= \left(n + \frac{1}{2}\right) \ln\left(\frac{n+1}{n}\right) - 1 \geq 0$$

D'où (v_n) est croissante.

$$w_{n+1} - w_n = v_{n+1} - v_n + \frac{1}{12n} - \frac{1}{12(n-1)}$$

$$= v_{n+1} - v_n + \frac{1}{12} - \frac{1}{n(n-1)}$$

$$= \left(n + \frac{1}{2}\right) \ln\left(\frac{n+1}{n}\right) - 1 + \frac{1}{12} - \frac{1}{n(n-1)}$$

$$\leq \frac{1}{12n^2} - \frac{1}{12n(n-1)} \leq \frac{1}{12} \left(\frac{1}{n^2} - \frac{1}{n^2 - n}\right) \leq 0$$

D'où (w_n) est décroissante.

$$\lim_{n \rightarrow +\infty} v_n - w_n = \lim_{n \rightarrow +\infty} \frac{1}{12(n-1)} = 0$$

Ainsi les suites (v_n) et (w_n) sont adjacentes.

$$\text{b) On pose } l = \lim_{n \rightarrow +\infty} v_n = \lim_{n \rightarrow +\infty} w_n.$$

$$\begin{aligned}
 2v_n - v_{2n} &= 2 \ln(n^n \sqrt{n} e^{-n}) - 2 \ln(n!) \\
 &= \ln((2n)^{2n} \sqrt{2n} e^{-2n}) + \ln((2n)!) \\
 &= \ln\left(\frac{n^{2n} n e^{-2n}}{(n!)^2} \frac{(2n)!}{(2n)^{2n} \sqrt{2n} e^{-2n}}\right) \\
 &= \ln\left(\frac{n(2n)!}{(n!)^2 2^{2n} \sqrt{2n}}\right) \\
 &= \ln\left(\frac{\sqrt{n} (2n)!}{(n!)^2 2^{2n} \sqrt{2}}\right)
 \end{aligned}$$

D'autre part :

$$\ln\left(\frac{F_{2n+1} \sqrt{2n}}{\pi}\right) = \ln\left(\frac{(2n)!}{2^{2n} (n!)^2} \frac{\sqrt{2n} \pi}{\pi} \frac{1}{2}\right) = \ln\left(\frac{(2n)! \sqrt{n}}{2^{2n} (n!)^2 \sqrt{2}}\right)$$

$$\text{D'où } 2v_n - v_{2n} = \ln\left(\frac{F_{2n+1} \sqrt{2n}}{\pi}\right)$$

$$\text{On a } F_{2n+1} \sim \sqrt{\frac{\pi}{2(2n+1)}} \Rightarrow \frac{F_{2n+1} \sqrt{2n}}{\pi} \sim \sqrt{\frac{\pi}{4n}} \frac{\sqrt{2n}}{\pi} = \frac{1}{\sqrt{2\pi}}$$

$$\text{Ainsi } \lim_{n \rightarrow +\infty} 2v_n - v_{2n} = \frac{1}{\sqrt{2\pi}} \Rightarrow 2l - l = \frac{1}{\sqrt{2\pi}} \Rightarrow l = \frac{1}{\sqrt{2\pi}}$$

$$\text{Alors } v_n \xrightarrow[n \rightarrow +\infty]{l = \frac{1}{\sqrt{2\pi}}} \ln\left(\frac{n^n \sqrt{n} e^{-n}}{n!}\right) \xrightarrow[n \rightarrow +\infty]{} \frac{1}{\sqrt{2\pi}}$$

$$\Leftrightarrow \frac{n^n \sqrt{n} e^{-n}}{n!} \sim \frac{1}{\sqrt{2\pi}} \Leftrightarrow n! \sim \sqrt{2\pi} n^n \sqrt{n} e^{-n} \sim \sqrt{2\pi n} n^n e^{-n}$$