

Problème: "Exponentiel"

Partie I.

$$\text{J. } \|A^p\| \leq \|A\|^p, \forall p \in \mathbb{N}^*.$$

$$0 \leq \frac{\|A^p\|}{p!} \leq \frac{\|A\|^p}{p!}$$

$$\text{on } \sum_{p \geq 0} \frac{\|A\|^p}{p!} \text{ C.V. vers } e^{\|A\|}$$

$$\Rightarrow \sum_{p \geq 0} \frac{\|A^p\|}{p!} \text{ C.V. } \Leftrightarrow \sum_{p \geq 0} \frac{A^p}{p!} \text{ est abs C.V.}$$

$$\text{on pose. } \exp(A) = \sum_{p=0}^{\infty} \frac{A^p}{p!} \Rightarrow \text{C.V. de } \exp(A)$$

$$2) A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}; B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$A^2 = 0_2 \text{ et } B^2 = 0_2$$

$$\Rightarrow \exp(A) = I_2 + A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$\exp(B) = I_2 + B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

$$\Rightarrow \exp(A) \cdot \exp(B) = \begin{pmatrix} e & 1 \\ 1 & 1 \end{pmatrix}$$

Requis: A est nilép d'indice $p \in \mathbb{N}^*$.

$$\Rightarrow \exp(A) = \sum_{k=0}^{p-1} \frac{A^k}{k!}$$

$$\text{soit } C = A+B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$C^2 = I_2 \Rightarrow C^{2p} = I_2$$

$$C^{2p+1} = I_2 \cdot C = C$$

$$\Rightarrow \exp(x) = \sum_{p=0}^{\infty} \frac{x^p}{p!} = \lim_{N \rightarrow \infty} \sum_{p=0}^N \frac{x^p}{p!} \quad (\text{car } \Sigma \text{ cv})$$

$$= \lim_{N \rightarrow \infty} \sum_{p=0}^N \frac{x^p}{(p)!} + \sum_{p=0}^{N-1} \frac{x^{p+1}}{(p+1)!}$$

$$= \lim_{N \rightarrow \infty} \sum_{p=0}^N \frac{1}{(p)!} I_2 + \sum_{p=0}^{N-1} \frac{1}{(p+1)!} C.$$

$$\Rightarrow \exp(x) = \cosh(x) I_2 + \sinh(x) C.$$

$$\Rightarrow \exp(A+B) = \begin{pmatrix} \cosh I & \sinh I \\ \sinh I & \cosh I \end{pmatrix} \neq \exp(A) \cdot \exp(B).$$

Th.

§. $AB=BA \Rightarrow \exp(A+B) = \exp(A) \cdot \exp(B)$

Prop.

$$\cosh x = \frac{e^x + e^{-x}}{2}.$$

$$= \frac{1}{2} \left(\sum_{n=0}^{\infty} \frac{x^n}{n!} + \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} \right)$$

$$= \frac{1}{2} \left(\sum_{n=0}^{\infty} \frac{1+(-1)^n}{n!} x^n \right)$$

$$\Rightarrow \cosh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}, \quad \forall x \in \mathbb{R}.$$

et

$$\sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}, \quad \forall x \in \mathbb{R}$$

a) Sei $P \in GL_n(\mathbb{R})$
 $f: \Pi_n(\mathbb{R}) \rightarrow \Pi_n(\mathbb{R})$
 $\pi \mapsto f(\pi) = P\pi P^{-1}$

f. lineäre

$\det \Pi_n(\mathbb{R}) \xrightarrow{\sim} \Rightarrow$ fest continue
 $\text{mer } \Pi_n(\mathbb{R})$

b) A ist diagonalisierbar.

$\Leftrightarrow \exists P \in GL_n(\mathbb{R})$ s. $A = P D P^{-1}$
 $\Rightarrow \forall k \in \mathbb{N}; A^k = P D^k P^{-1}$

$\Rightarrow \exp(A) = \sum_{k=0}^{\infty} \frac{A^k}{k!} = \sum_{k=0}^{\infty} \frac{P D^k P^{-1}}{k!}$
 $= \lim_{N \rightarrow \infty} \left(\sum_{k=0}^N \frac{P D^k P^{-1}}{k!} \right)$

$= \lim_{N \rightarrow \infty} \left(P \underbrace{\sum_{k=0}^N \frac{D^k}{k!}}_{= e^D} P^{-1} \right)$

$= \lim_{N \rightarrow \infty} f(S_N)$

$= f\left(\lim_{N \rightarrow \infty} S_N\right)$ can fest continue?

$= P \left(\lim_{N \rightarrow \infty} S_N \right) P^{-1}$

$= P \sum_{k=0}^{\infty} \frac{D^k}{k!} P^{-1}$

$\Rightarrow \exp(A) = P(\exp D) P^{-1}$

$\Rightarrow$ s. A ist diagonalisierbar.

$\Rightarrow \exp(A)$

$\Rightarrow \det_{\mathbb{R}}(\exp(A)) = \prod e^{\lambda_j} = e^{\sum \lambda_j} = e^{\text{tr}(A)}$

Question

Soit $A \in \mathcal{M}_n(\mathbb{R})$.

Pq. si dvp de A , alors e'est une v.p. de $\exp(A)$

Réponse

ona dvp de $A \Rightarrow \exists X \neq 0$ tq $AX = \lambda X$.

$\Rightarrow \forall n \in \mathbb{N}; A^n X = \lambda^n X$

$\Rightarrow (\exp A) \cdot X = \left(\sum_{n=0}^{\infty} \frac{A^n}{n!} \right) \cdot X$

$$= \sum_{n=0}^{\infty} \frac{(A^n X)}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{\lambda^n X}{n!} = e^{\lambda} X$$

$g: \mathcal{M}_n(\mathbb{R}) \rightarrow \mathcal{M}_n(\mathbb{R})$
 $\pi \mapsto \pi X$
Continue

Question

$$(\|A\| \leq \|A\| \cdot \|B\|)$$

Soit $A \in \mathcal{M}_n(\mathbb{R})$ tq $\|A\| < 1$

Pq a) $\sum_{n=0}^{\infty} A^n$ c.v.

b) Pq $I - A$ est inversible et q

$$(I - A)^{-1} = \sum_{n=0}^{\infty} A^n$$

$$(I - A) \cdot \sum_{n=0}^N A^n = I - A^{N+1}$$

$\downarrow N \rightarrow \infty$

$$(I - A) \cdot \sum_{n=0}^{\infty} A^n = I$$

$$5) \varphi: \mathbb{R}_{p-1}[X] \rightarrow \mathbb{R}^p$$

$$P \mapsto \varphi(P) = (P(d_1), \dots, P(d_p)).$$

a) φ est linéaire

$$\text{Soit } P \in \ker \varphi \Leftrightarrow \begin{cases} \varphi(P) = 0_{\mathbb{R}^p} \\ \text{et } P \in \mathbb{R}_{p-1}[X] \end{cases}$$

$$\Rightarrow \begin{cases} P(d_i) = 0, \forall i \in \{1, \dots, p\} \\ \deg P \leq p-1 \end{cases}$$

$$\Rightarrow \left. \begin{array}{l} P \text{ admet } p \text{ racines distinctes} \\ \deg P \leq p-1 \end{array} \right\} \Rightarrow P = 0.$$

$\Rightarrow \varphi$ injective.

$$\text{et on a } \dim(\mathbb{R}_{p-1}[X]) = \dim \mathbb{R}^p = p \Rightarrow$$

$\Rightarrow \varphi$ est bijective.

$\Rightarrow \varphi$ isomorphe.

b) on a φ surjective.

$$\text{Pour } (e^{d_1}, \dots, e^{d_p}) \in \mathbb{R}^p, \exists L \in \mathbb{R}_{p-1}[X]$$

$$\text{t.q. } \varphi(L) = (e^{d_1}, \dots, e^{d_p})$$

$$\Leftrightarrow L(d_i) = e^{d_i}, \forall i \in \{1, \dots, p\}.$$

c) $B = (L_1, \dots, L_p)$ base de $\mathbb{R}_{p-1}[X]$.

$$\text{d) on a } L \in \mathbb{R}_{p-1}[X] \Rightarrow$$

$$L(X) = \sum_{i=1}^p d_i L_i(X).$$

$$\text{or } L\left(\underbrace{d_j}_{e^{d_j}}\right) = \sum_{i=1}^p d_i \underbrace{L_i(d_j)}_{\delta_{ij}} = d_j.$$

$$\Rightarrow L(X) = \sum_{i=1}^p e^{d_i} L_i(X).$$

7) A est diagonalisable.
 $d_1, \dots, d_p$ sont les vp de A et e_i

a) on a. $A = P D P^{-1}$.

$$\begin{aligned} \Rightarrow \exp(A) &= P \exp D P^{-1} \quad (4-6) \\ &= P \text{diag}(e^{d_1}, \dots, e^{d_p}) P^{-1} \\ &= P \text{diag}(L(d_1), \dots, L(d_p)) P^{-1} \\ &= L(P \text{diag}(d_1, \dots, d_p) P^{-1}) \\ \Rightarrow \exp(A) &= L(A) \end{aligned}$$

b) $f \in \mathcal{L}(\mathbb{R}^n)$ et $\text{Trat}(f, B_C) = A$.
 on $E_i = \ker(f - d_i \text{id}_{\mathbb{R}^n})$
 $P_i = L_i(f) \in \mathcal{L}(E_i)$

on a. A est diagonalisable $\Leftrightarrow f$ est diagonalisable

$$\Rightarrow \mathbb{R}^n = E = \bigoplus_{k=1}^p E_k = E_i \oplus \bigoplus_{\substack{k=1 \\ k \neq i}}^p E_k$$

$\forall x \in E, \exists y \in E_i$ et $z \in \bigoplus_{\substack{k=1 \\ k \neq i}}^p E_k$

tg. $x = y + z$, mg. $P_i(x) = y$.

on $P_i(x) = P_i(y) + P_i(z)$

or $y \in E_i = \ker(f - d_i \text{id}) \Leftrightarrow f(y) = d_i y$.

$$\Rightarrow (L_i(f))(y) = L_i(d_i) y = y$$

$$\Rightarrow P_i(y) = y$$

$$\exists z \in \bigoplus_{\substack{k=1 \\ k \neq i}}^p E_k \Rightarrow z = \sum_{\substack{k=1 \\ k \neq i}}^p z_k \text{ avec } z_k \in E_k$$

$$\begin{aligned} \Rightarrow L_i(f)(z) &= \sum_{\substack{k=1 \\ k \neq i}}^p L_i(f)(z_k) \\ &= \sum_{\substack{k=1 \\ k \neq i}}^p L_i(d_k) z_k \end{aligned}$$

$$L_i(d_k) = d_k \text{id}_{E_i}$$

$$\geq \frac{1}{n} \cdot 1$$

$$c) \exp(A) = L(A)$$

$$= \sum_{i=1}^p e^{d_i} L_i(A)$$

$$= \sum_{i=1}^p e^{d_i} L_i(\text{Trat}(f, B))$$

$$= \sum_{i=1}^p e^{d_i} \text{Trat}(L_i(f), B_i)$$

$$= \sum_{i=1}^p e^{d_i} \text{Trat}(p, B)$$

$$\begin{aligned} A &= \text{Trat}(f, B) \\ &= \text{Trat}(f^n, B) = A^n = (\text{Trat}(f, B))^n \\ &= \text{Trat}(P(f), B) = P(A) = P(\text{Trat}(f, B)) \end{aligned}$$

$$\forall P \in (K[X])$$

Partie II :

$$\begin{aligned} 1) \quad U \in \mathcal{L}(E) \quad \left\{ \begin{array}{l} \text{et } \dim E < \infty \end{array} \right\} &\Rightarrow U \text{ est continue.} \\ \Leftrightarrow \exists T \geq 0 \text{ tel que } \|Ux\| &\leq T \|x\|, \forall x \in E. \end{aligned}$$

$$\Rightarrow \forall x \in E, x \neq 0, \quad \frac{\|Ux\|}{\|x\|} \leq T.$$

$$\Rightarrow \sup_{x \neq 0} \frac{\|Ux\|}{\|x\|} \text{ est fini.}$$

$$\text{et } \sup_{\|x\|=1} \|Ux\| \text{ existe.}$$

Pr. $\sup_{x \neq 0} \frac{\|U(x)\|}{\|x\|} = \sup_{\|x\|=1} \|U(x)\|$

ona.

si $\|x\|=1 \Rightarrow \frac{\|U(x)\|}{\|x\|} \leq \sup_{x \neq 0} \frac{\|U(x)\|}{\|x\|}$
 $\Rightarrow \frac{\|U(x)\|}{\|x\|} \leq \sup_{x \neq 0} \frac{\|U(x)\|}{\|x\|}$

$\Rightarrow \sup_{\|x\|=1} \|U(x)\| \leq \sup_{x \neq 0} \frac{\|U(x)\|}{\|x\|}$ (1)

soit $x \in E \setminus \{0\} \Rightarrow y = \frac{x}{\|x\|}$ verif. $\|y\|=1$

$\Rightarrow \frac{\|U(y)\|}{\|y\|} \leq \sup_{\|z\|=1} \|U(z)\|$

$\Rightarrow \frac{\|U(x)\|}{\|x\|} \leq \sup_{\|z\|=1} \|U(z)\|$

$\Rightarrow \sup_{x \neq 0} \frac{\|U(x)\|}{\|x\|} \leq \sup_{\|z\|=1} \|U(z)\|$ (2)

$\Rightarrow \sup_{x \neq 0} \frac{\|U(x)\|}{\|x\|} = \sup_{\|z\|=1} \|U(z)\|$

* Pr. $U \rightarrow \|U\| = \sup_{\|x\|=1} \|U(x)\|$

Pr. $\|\cdot\|$ est une norme sur $\mathcal{L}(E)$

si $\|U\|=0 \Leftrightarrow \sup_{\|x\|=1} \|U(x)\|=0$

$\Rightarrow \|U(x)\|=0$ si $\|x\|=1$

$U(x)=0, \forall x \in S(0,1)$

$\Rightarrow \forall x \neq 0, \frac{x}{\|x\|} \in S(0,1)$

$\Rightarrow \frac{U(x)}{\|x\|} = 0 \Rightarrow U(x)=0, \forall x \in E$
 $\Rightarrow U=0$

$$5) \pi_U(x) = (x-1)^2(x-2)$$

1) π_U n'est pas une norme simple.

$\Rightarrow U$ n'est pas diagonalisable.

$$2) E = \ker(U - \text{id}_E) \oplus \ker(U - 2\text{id}_E)$$

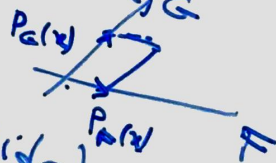
$$3) p = (U - \text{id}_E)^2, \quad q = U(2\text{id}_E - U)$$

$$a) p + q = \text{id}_E$$

b) $\pi_q p$: proj sur $\ker(U - 2\text{id}_E) // \ker(U - \text{id}_E)^2$.

$$\forall x \in E; x = x_1 + x_2,$$

$$x_1 \in \ker(U - \text{id}_E)^2 \text{ et } x_2 \in \ker(U - 2\text{id}_E).$$



$$\pi_q. p(x) = x_2$$

$$\text{Donc } p(x) = p(x_1 + x_2) = p(x_1) + p(x_2).$$

$$\text{Donc } x_2 \in \ker(U - \text{id}_E)^2.$$

$$\text{Donc } (U - \text{id}_E)^2(x_2) = 0$$

$$p(x_1) = 0$$

$$x_2 \in \ker(U - 2\text{id}_E)$$

$$(U - 2\text{id}_E)(x_2) = 0$$

$$p(x_2) = (\text{id}_E - q)(x_2)$$

$$= x_2 - q(x_2)$$

$$\Rightarrow p(x) = x_2$$

$$* \text{ sur } q. q = \text{id}_E - p$$

$$\Rightarrow q \text{ projection sur } \ker(U - \text{id}_E)^2 // \text{à } \ker(U - 2\text{id}_E).$$

$$c) \forall x \in E; \Rightarrow p(x) \in \text{Im } p = \ker(U - 2\text{id}_E)$$

$$\Rightarrow U(p(x)) = 2p(x)$$

$$\text{réciproque } U^k \circ p = 2^k p$$

$$\begin{aligned}
 b) \exp(u) \circ p &= \left(\sum_{n=0}^{\infty} \frac{u^n}{n!} \right) \circ p. \quad \varphi: f \rightarrow f \circ p \\
 &= \sum_{n=0}^{\infty} \left(\frac{u^n}{n!} \circ p \right). \quad \text{Continuity} \\
 &= \sum_{n=0}^{\infty} \frac{u^n}{n!} \cdot p = e^u \cdot p
 \end{aligned}$$

$$\begin{aligned}
 5.) \quad x^k &= (x-1)^e \varphi(x) + ax + b \\
 \xrightarrow{x=1} \quad 1 &= 0 + b \\
 \Rightarrow k x^{k-1} &= e(x-1)\varphi(x) + (x-1)^e \varphi'(x) + a \\
 \xrightarrow{x=1} \quad \boxed{k=a} \Rightarrow b &= 1-k \\
 \Rightarrow x^k &= (x-1)^e \varphi(x) + kx + (1-k)
 \end{aligned}$$

$$\begin{aligned}
 b) \exp(u) \circ q &= \sum_{n=0}^{\infty} \left(\frac{u^n}{n!} \circ q \right) \quad (\text{Continuity}) \\
 \text{or then} \quad u^n \circ q &= (u - \text{id}_E)^e \circ \varphi(u) + n \cdot u + (1-n) \cdot \text{id}_E \circ q \\
 &= (u - \text{id}_E)^e \circ q \circ \varphi(u) + n u \circ q + (1-n)q \\
 &= \frac{(u - \text{id}_E)^e (e \text{id}_E - u) \circ u \circ \varphi(u) + \dots}{- \frac{1}{e} \frac{1}{u}} \\
 &= n u \circ q + (1-n)q \\
 \Rightarrow \exp(u) \circ q &= \sum_{n=0}^{\infty} \frac{u^n}{n!} \circ q + \sum_{n=0}^{\infty} \frac{(1-n)}{n!} q \\
 &= \sum_{n=1}^{\infty} \frac{1}{(n-1)!} u \circ q + (e - e/q) \\
 &= e u \circ q
 \end{aligned}$$

$$\begin{aligned}
 \text{c) } \exp U &= \exp U_{\text{ide}} \\
 &= \exp U_0(p+q) \\
 &= \exp U_0 p + \exp U_0 q \\
 &= e^p + e \cdot U_0 q
 \end{aligned}$$