

$$\int_I \sum_{n=0}^{\infty} p_n(x) dx = \sum_{n=0}^{\infty} \int_I p_n(x) dx$$

si $I = [a, b]$.

Continu
+
C15

T.I.T.T.

$\forall n \in \mathbb{N}$: f_n cont. intégrable.

$\sum_{n=0}^{\infty} p_n \subset S \rightarrow S$ sur I

S est C.T. sur I

$\sum_{n=0}^{\infty} \int_I |p_n(x)| dx < \infty$ calculer ou majorer

Alors S est intégrable sur I
 $\int \sum = \sum \int$

I intervalle $a < b$

Appliquer T.C.D
à (f_n)

$$\int_I S(x) dx = \int_I \lim_{n \rightarrow \infty} S_n(x) dx$$

$$\stackrel{\text{T.C.D}}{\Rightarrow} \lim_{n \rightarrow \infty} \int_I S_n(x) dx$$

$$= \lim_{n \rightarrow \infty} \sum_{k=0}^n \int_I p_k(x) dx$$

$$\Rightarrow \sum_{k=0}^{\infty} \int_I p_k(x) dx = \int_I S(x) dx$$

$$\int_I S(x) dx =$$

$$\int_I S_n(x) + R_n(x) dx$$

$$= \int_I S_n(x) dx + \int_I R_n(x) dx$$

$$= \sum_{k=0}^n \int_I p_k(x) dx + \int_I R_n(x) dx$$

$$\text{Pq. } \int_I R_n(x) dx \xrightarrow{n \rightarrow \infty} 0$$

$$\Rightarrow \sum_{k=0}^{\infty} \int_I p_k(x) dx = \int_I S(x) dx$$

Exercice F

1) $\sum_{n \geq 1} \frac{1}{n^x}$ est une série de Riemann, alors
 de CV SSA $x > 1$.

$$\Rightarrow D_f =]1, \infty[.$$

2) $F(x) = \int_0^{\infty} \frac{t^{x-1}}{e^t - 1} dt$
 $\forall x > 1$, $t \mapsto \frac{t^{x-1}}{e^t - 1}$ est C.P. sur $]0, \infty[$.
Asymptote $f(t) \sim \frac{t^{x-1}}{t} = \frac{1}{t^{2-x}}$ qui est intégrable
car $(x > 1) \Rightarrow f(t) = \frac{t^{x-1}}{e^t - 1} \sim \frac{t^{x-1}}{t} = \frac{1}{t^{2-x}}$ (pour $t \rightarrow \infty$)
 $\Rightarrow f(t) = o\left(\frac{1}{t^2}\right)$ (à $+\infty$)
 $\Rightarrow F(x)$ existe, $\forall x > 1$.

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3) b) $\forall x > 1$: $F(x) = \Gamma(x) \cdot \xi(x)$

$$\int_0^{\infty} \frac{t^{x-1}}{e^t - 1} dt = \Gamma(x) \sum_{n=0}^{\infty} \frac{1}{n^x}$$

or $\forall x > 1$: $F(x) = \int_0^{\infty} \frac{t^{x-1}}{e^t - 1} dt$

$$= \int_0^{\infty} \frac{t^{x-1} e^{-t}}{1 - e^{-t}} dt = \int_0^{\infty} t^{x-1} e^{-t} \sum_{n=0}^{\infty} (\bar{e}^{-t})^n dt$$

$$= \int_0^{\infty} \sum_{n=1}^{\infty} t^{x-1} \bar{e}^{-nt} dt \quad (x \bar{e}^{-t} < 1)$$

$$= \int_0^{\infty} \sum_{n=1}^{\infty} f_n(t) dt \quad \text{and} \quad f_n(t) = t^{x-1} \bar{e}^{-nt}$$

$$\stackrel{(*)}{=} \int_0^{\infty} \sum_{n=1}^{\infty} f_n(t) dt$$

$$= \sum_{n=1}^{\infty} \int_0^{\infty} t^{x-1} \bar{e}^{-nt} dt \quad u = nt$$

$$= \sum_{n=1}^{\infty} \int_0^{\infty} \left(\frac{u}{n}\right)^{x-1} \bar{e}^{-u} \frac{du}{n}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^x} \cdot \Gamma(x) = \Gamma(x) \cdot \xi(x)$$

Justification $\circledast$
 • then $\{f_n\}$ is C.T. and integrable on $[0, \infty)$
can $\left\{ \begin{array}{l} f_n(t) = o\left(\frac{1}{n^x}\right) \\ f_n(t) \rightarrow 0 \text{ as } n \rightarrow \infty \end{array} \right.$

$\sum_{n=1}^{\infty} f_n(t) \in C$ for $x > 1$
 Set $C.T.$ on $[0, \infty)$
 $f(t) = \sum_{n=1}^{\infty} \frac{t^{x-1}}{e^{nt}} = S(t)$

$$\sum_{n \geq 1} \int_0^{\infty} |f_n(t)| dt = \sum_{n \geq 1} \frac{1}{n^2} \cdot \Gamma(x) \cdot C_V(x, 1)$$

Donc la permutation (*) est justifiée par le T.I.T.T.

4) D'après 3/ pour $x=2$.

$$\int_0^{\infty} \frac{t}{e^t - 1} dt = F(2) = \Gamma(2) \cdot \xi(2) \\ = \left(\sum_{n=0}^{\infty} \frac{1}{n^2} \right) \cdot \Gamma(2)$$

$$\text{or } \Gamma(2) = \Gamma(1) \quad (\Gamma(x+1) = x\Gamma(x)) \\ = \int_0^{\infty} e^{-t} dt = 1.$$

$$\Rightarrow \int_0^{\infty} \frac{t}{e^t - 1} dt = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

$$5/a) \quad \int_0^{\infty} \frac{ue^u}{e^{ue} - 1} du = \int_0^{\infty} \frac{t}{(e^t - 1)e^{\sqrt{t}}} \frac{dt}{2\sqrt{t}} \\ t = u^2, \quad u = \sqrt{t} \Rightarrow du = \frac{1}{2\sqrt{t}} dt$$

$$= \frac{1}{2} \int_0^{\infty} \frac{t^{\frac{1}{2}}}{(e^t - 1)} dt = \frac{1}{2} F\left(\frac{3}{2}\right)$$

$$= \frac{1}{2} \Gamma\left(\frac{3}{2}\right) \cdot \xi\left(\frac{3}{2}\right)$$

$$\text{or } \Gamma\left(\frac{3}{2}\right) = \frac{1}{2} \Gamma\left(\frac{1}{2}\right) = \frac{1}{2} \sqrt{\pi}$$

$$\Rightarrow \int_0^{\infty} \frac{ue^u}{e^{ue} - 1} du = \frac{\sqrt{\pi}}{2} \xi\left(\frac{3}{2}\right)$$

$$b) \int_0^{\infty} \left(\frac{\ln(1+u)}{u} \right)^2 du = 2\zeta(2)$$

$$f = (\ln(1+u))^2 \rightarrow f' = 2 \frac{\ln(1+u)}{1+u}$$

$$g' = \frac{1}{u^2} \rightarrow g = -\frac{1}{u}$$

$$\lim_{u \rightarrow 0} -\frac{(\ln(1+u))^2}{u} = 0 \quad (\ln(1+u) \sim u)$$

$$\lim_{u \rightarrow \infty} -\frac{(\ln(1+u))^2}{u} = 0$$

$$\text{or } \int_0^{\infty} \left(\frac{\ln(1+u)}{u} \right)^2 du < \infty$$

$$\lim_{u \rightarrow 0} \frac{(\ln(1+u))^2}{u} = 1$$

$$\frac{(\ln(1+u))^2}{u^2} = \frac{(\ln(1+u))^2}{u^2}$$

$$\Rightarrow \left(\frac{\ln(1+u)}{u^2} \right)^2 \underset{u \rightarrow \infty}{=} 0 \left(\frac{1}{u^2} \right)$$

$$\text{Donc } I = 2 \int_0^{\infty} \frac{\ln(1+u)}{u(1+u)} du.$$

$$x = \ln(1+u) \\ dx = \frac{1}{1+u} du.$$

$$\Rightarrow I = 2 \int_0^{\infty} \frac{x}{e^x - 1} dx = 2\Gamma(2) = 2\zeta(2).$$

$$c) \pi \zeta(2) = \frac{1}{2} \cdot \frac{1}{2k}$$

$$\text{or } \zeta(2) = \sum_{k=1}^{\infty} \frac{1}{k^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots$$

$$\pi_q \lim_{x \rightarrow \infty} \sum_{n=3}^x \frac{2^n}{n^x} = 0.$$

• Sei $q_n(x) = \frac{2^n}{n^x} : n \geq 3.$

• $n \geq 3; \lim_{x \rightarrow \infty} q_n(x) = \lim_{x \rightarrow \infty} \left(\frac{2}{n}\right)^x = 0$

• Sei $a > 1$

$$\sup_{x \in [0, \infty)} |q_n(x)| = \left(\frac{2}{n}\right)^a \quad (x \mapsto \left(\frac{2}{n}\right)^x \text{ ist } \downarrow)$$

$$\sum_{n \geq 3} \left(\frac{2}{n}\right)^a = 2^a \cdot \sum_{n \geq 3} \frac{1}{n^a} < \infty \quad (a > 1)$$

$$\Rightarrow \sum_{n \geq 3} q_n \subset \mathbb{N} \Rightarrow \subset \cup_{n \in \mathbb{N}} [0, \infty)$$

Diri:

$$\lim_{x \rightarrow \infty} \sum_{n=3}^x \left(\frac{2}{n}\right)^x = \sum_{n=3}^{\infty} \lim_{x \rightarrow \infty} \left(\frac{2}{n}\right)^x = 0$$

Ming:

$$2^x \left(\frac{2}{3}\right)^{x-1} \xrightarrow{x \rightarrow \infty} 0 \quad \text{u.} \quad \left| \sum_{n=3}^x \left(\frac{2}{n}\right)^{x-1} \right| \leq \frac{1}{3^{x-1}}$$

1) On a. $\xi(x) - 1 \sim_{x \rightarrow \infty} \frac{1}{2x}$

$\Rightarrow \Gamma(x) (\xi(x) - 1) \sim_{x \rightarrow \infty} \frac{\Gamma(x)}{2x}$

||

$F(x) - \Gamma(x)$

$\int_0^{\infty} \left(\frac{t^{x-1}}{e^t - 1} - t^{x-1} e^{-t} \right) dt = \int_0^{\infty} \frac{t^{x-1}}{e^t (e^t - 1)} dt$

$\sum_{n \geq 1} \frac{1}{n^x} \xrightarrow{x \rightarrow 1^+} \infty$

Cor.

$\forall n \geq 1: \lim_{x \rightarrow \pi} \frac{1}{n^x} = \frac{1}{n}$

et $\sum_{n \geq 1} \frac{1}{n}$ div. $\Rightarrow \sum_{n \geq 1} \frac{1}{n^x} \xrightarrow{x \rightarrow 1^+} \infty$

$\lim_{x \rightarrow \pi} \xi(x) = +\infty$

On a. ξ est décroissante sur $J(1, \infty)$

$\forall x, y \in J(1, \infty)$, $x \leq y$. $\frac{1}{n^x} \geq \frac{1}{n^y}$, $\forall n \geq 1$.

Donc $\lim_{x \rightarrow \pi} \xi(x) = \lim_{x \rightarrow \pi} \sum_{n=1}^{\infty} \frac{1}{n^x} = \sum_{n=1}^{\infty} \frac{1}{n}$

Supposons $\lim_{x \rightarrow \pi} \xi(x) = p$

$$\forall x > 1; \quad \zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x} \geq \sum_{n=1}^N \frac{1}{n^x}.$$

$$\Rightarrow \lim_{x \rightarrow 1+} \zeta(x) \geq \sum_{n=1}^N \frac{1}{n}$$

$$\forall \ell < \infty.$$

$$\Rightarrow \forall N \geq 1; \quad \sum_{n=1}^N \frac{1}{n} \leq \ell.$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n} \text{ converge. absolutely.}$$

$$\Rightarrow \lim_{x \rightarrow 1+} \zeta(x) = \infty.$$

Problem.

$f:]0, \infty[\rightarrow \mathbb{R}_+$, Continuous et $\searrow$

$$\lim_{x \rightarrow \infty} f(x) = 0$$

$$x > 0; \quad U_k(x) = f(x+k) - \int_{x+k}^{x+k+1} f(t) dt$$

1) a) $0 \leq U_k(x) \leq f(x+k) - f(x+k+1)$

b) or $\sum_{k \geq 0} f(x+k) - f(x+k+1) \subset V$ con.

$$\Rightarrow \sum_{k \geq 0} U_k(x) \subset V; \quad \forall x > 0 \Leftrightarrow \lim_{k \rightarrow \infty} f(x+k) = 0$$

on pose. $U(x) = \sum_{k=0}^{\infty} U_k(x), \quad x > 0$

4) a) $f(x) = e^{-x}$

est continue, décroissante et.

$$\lim_{x \rightarrow \infty} f(x) = 0$$

Alors $U(x) = \sum_{k=0}^{\infty} U_k(x)$ existe.

$$U_k(x) = e^{-(x+k)} - \int_{x+k}^{x+k+1} e^{-t} dt.$$

$$= e^{-(x+k+1)}$$

$$\Rightarrow U(x) = \sum_{k=0}^{\infty} e^{-k} \cdot e^{-x} = e^{-x} \cdot \frac{e^{-1}}{1 - e^{-1}}$$

$$\Rightarrow U(x) = \frac{e^{-x}}{e-1}, \quad \forall x > 0.$$

$$3) R_n(x) = \sum_{k=n+1}^{\infty} U_k(x)$$

$$\forall x > 0; 0 \leq \sum_{k=n+1}^N U_k(x) \leq \sum_{k=n+1}^N (f(x+k) - f(x+k-1))$$

$$f(x+n+1) - f(x+N+1)$$

$$\text{or } f(x+N+1) \xrightarrow[n \rightarrow \infty]{} 0$$

$$\text{Donc } 0 \leq R_n(x) \leq f(x+n+1), \forall x > 0$$

4) a). $\forall k \in \mathbb{N}$; $t \mapsto f(t)$ est continue sur $J =]0, \infty[$.
 $x \mapsto x+k$ et $x \mapsto x+k-1$ sont continus.

Donc $x \mapsto \int_{x+k-1}^{x+k} f(t) dt$ est continue sur J .

$\Rightarrow U_k$ est continue sur J .

D'après

$$0 \leq R_n(x) \leq f(x+n+1) \leq f(n+1) \quad \forall x > 0$$

$$\Rightarrow 0 \leq \sup_{x \in J} R_n(x) \leq f(n+1) \xrightarrow[n \rightarrow \infty]{} 0$$

$$\Rightarrow R_n \xrightarrow[n \rightarrow \infty]{} 0 \text{ sur } J$$

$$\Rightarrow \sum_{n \geq 0} u_n = \lim_{j \rightarrow \infty} \sum_{n \geq 0} u_n^{(j)}$$

Disc: Vert Continuum 30, 200.

$$b) \lim_{x \rightarrow a} u(x) = \lim_{x \rightarrow a} \sum_{n=0}^{\infty} u_n(x)$$

$$\lim_{x \rightarrow \infty} U_n(x) = 0 \quad (\text{für } n \geq 1)$$

$$1. \sum_{n \geq 0} U_n C(n) \text{ is Jordan}$$

Akon $\lim_{n \rightarrow \infty} U_n(x) = \sum_{n=0}^{\infty} \lim_{x \rightarrow 0} U_n(x) = 0$
Auswertung:

Act 1

$$\Rightarrow \lim_{n \rightarrow \infty} U(x) = \sum_{k=0}^{\infty} U_k(x) + R_n(x)$$

$$\Rightarrow \lim_{x \rightarrow 0} U(x) = \sum_{k=0}^{\infty} \lim_{x \rightarrow 0} U_k(x) + \lim_{x \rightarrow 0} R_n(x)$$

5) $\lim_{0^+} f(x) = +\infty$

a) $\exists \epsilon \in \mathbb{R}^+ \mid \dots$

$$\int_x^{x+\epsilon} f(t) dt = \int_x^{x+\epsilon} f(t) dt + \int_{x+\epsilon}^{x+1} f(t) dt.$$

$$\leq f(x) \cdot \epsilon + f(x+\epsilon) \cdot \underbrace{(1-\epsilon)}_{< 1}$$

b) Req. li. $\lim_{x \rightarrow \infty} \frac{\int_x^{x+1} f(t) dt}{f(x)} = 0$

$$\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = 0$$

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = 0 \text{ con } f(x) \rightarrow 0 \text{ or } g(x) \rightarrow \infty$$

$$\Rightarrow \exists \delta > 0, \forall x \in]x_0, x_0 + \delta[, 0 < \frac{f(x) - f(x_0)}{f(x_0)} < \varepsilon$$

$$\Rightarrow \forall \varepsilon \in]0, 1[, \exists \delta > 0, \forall x \in]x_0, x_0 + \delta[; 0 < \frac{\int_x^{x+\delta} f(t) dt}{f(x)} \leq 2\varepsilon.$$

$$\Downarrow$$

$$\text{mon. } \forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall x \in]x_0, x_0 + \delta[$$

$$\Rightarrow \lim_{\delta \rightarrow 0} \frac{\int_x^{x+\delta} f(t) dt}{f(x)} = 0$$

$$f(x) > \varepsilon$$

$$\frac{1}{\varepsilon} \quad \frac{1}{\varepsilon} \quad \frac{1}{\varepsilon}$$

c) $U_k(x) = \frac{u_k(x)}{f(x)}$

$\sum_{k=0}^{\infty} U_k \subset S$ sur $]0, 1[$ vers $\frac{u}{f}$.

D'après 3/

$$0 \leq R_n(u) = \frac{R_n(u)}{f(x)} \leq \frac{f(x+n)}{f(x)} \leq \frac{f(x+1)}{f(x)}; \forall x \in]0, 1[$$

$$\Rightarrow \sup_{x \in]0, 1[} R_n(u) \leq \frac{f(x+1)}{f(x)} \xrightarrow{n \rightarrow \infty} 0$$

$$\Rightarrow R_n \subset U_n \text{ sur }]0, 1[$$

$$\Downarrow$$

$$\sum_{k=0}^{\infty} U_k \subset S \text{ sur }]0, 1[$$

d) $\pi q. \quad U \sim f \Leftrightarrow \pi q. \quad \lim_{\sigma \downarrow 0} \frac{U(x)}{f(x)} = 1.$

ona $\lim_{\sigma \downarrow 0} \frac{U(x)}{f(x)} = \lim_{\sigma \downarrow 0} \sum_{k=0}^{\infty} U_k(x)$

• $\sum_{k=0}^{\infty} U_k \subset W \text{ near } J_0 \cap I$

• Pour $k=0$; $\lim_{x \downarrow \sigma} U_0(x) = \lim_{\sigma \downarrow 0} \frac{f(x) - \int_x^{x+\sigma} f(t) dt}{f(x)} = 1$

• Pour $k \geq 1$;

$0 \leq \frac{U_k(x)}{f(x)} \leq \frac{f(x+k) - f(x+k+1)}{f(x)} \xrightarrow{x \downarrow \sigma} 0$

Alon. $\lim_{\sigma \downarrow 0} \frac{U(x)}{f(x)} = \sum_{k=0}^{\infty} \lim_{\sigma \downarrow 0} U_k(x) = 1 + 0 = 1$

Exercice 6.

$x > 0.$

$f(t) = \frac{e^{-tx}}{\sqrt{t}(1+e^t)}; \quad t > 0$

1) $f(t) = \frac{e^{-tx}}{\sqrt{t}} \sum_{n=0}^{\infty} (-e^t)^n (1 - e^t < 1)$
 $= \sum_{n=0}^{\infty} (-1)^n \frac{e^{-(n+x)t}}{\sqrt{t}}$

3) $\forall n \in \mathbb{N}, f_n$ est continue sur $J_0 \cap I$.

4) $\forall x > 0, \forall t > 0$

$\sum_{n=0}^{\infty} (-1)^n \frac{e^{-(n+x)t}}{\sqrt{t}}$

$\left(\frac{e^{-(n+x)t}}{\sqrt{t}} \right)_{n \geq 0}$ vérifie T.S.V.

$\lim_{t \downarrow 0} \frac{e^{-(n+x)t}}{\sqrt{t}} = 0$

Also: $|R_n(t)| \leq |p_{n+1}(t)|, \forall t > 0$

$$|\int_0^\infty R_n(t) dt| \leq \int_0^\infty |R_n(t)| dt \leq \int_0^\infty |p_{n+1}(t)| dt$$

$$\int_0^\infty |p_{n+1}(t)| dt = \int_0^\infty \frac{e^{-(n+1+x)t}}{\sqrt{t}} dt$$

$$u = \sqrt{n+1+x} \cdot \sqrt{t}$$

$$du = \frac{\sqrt{n+1+x}}{2\sqrt{t}} dt$$

$$\Rightarrow \left| \int_0^\infty R_n(t) dt \right| \leq \int_0^\infty \frac{2 e^{-u^2}}{\sqrt{n+1+x}} du = \frac{1}{\sqrt{n+1+x}} \cdot \int_0^\infty e^{-u^2} du$$

$$\Rightarrow \lim_{n \rightarrow \infty} \int_0^\infty R_n(t) dt = 0$$

5)

$$\int_0^\infty f(t) dt = \int_0^\infty S_n(t) + R_n(t) dt = \int_0^\infty S_n(t) dt + \int_0^\infty R_n(t) dt$$

$$= \sum_{k=0}^n \int_0^\infty p_k(t) dt + \int_0^\infty R_n(t) dt$$

monoblock

$$\int_0^\infty p(t) dt = \sum_{k=0}^\infty \int_0^\infty p_k(t) dt = \sum_{k=0}^\infty \frac{L^{(k)} \cdot \sqrt{\pi}}{\sqrt{n+1+x}}$$