

Exercício, 17 p 2017.

$$B_0 = 1;$$

$$B_{n+1} = \frac{1}{n+2} \sum_{k=0}^n C_{n+2}^k B_k$$

1) $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$

$$2) \sum_{k=0}^n C_{n+1}^k B_k = \sum_{k=0}^{n-1} C_{n+1}^k B_{k+1}$$

$$= \sum_{k=0}^{n-1} C_{n+1}^k B_k + (n+1)B_n = 0$$

3) Tq. $\forall n \in \mathbb{N}, |B_n| \leq n!$

- Für $n=0 \Rightarrow |B_d| = 2 \leq 0! = 1$
- Sei $n \in \mathbb{N}$. Nach Lemma 1.1:

• Sei $n \in \mathbb{N}$. $\Rightarrow |B_d| = 2 \leq |O| = 2$
 für tout $k \in \{0, \dots, n\}$ $\{e, \tau\}$
 und $1 \in$

On a. $|B_{k+1}| \leq \sum_{h=2}^n |C_{k+1}^h| |B_k|$

$$\approx \frac{1}{h+r} \sum_{k=0}^n C_{h+r}^k k! (H \cdot R)!$$

$$\leq \frac{1}{n+2} \cdot \sum_{k=0}^n \frac{(n+2)!}{(n+2-k)!}$$

$$\leq (n+1)! \sum_{p=2}^{n+2} \frac{1}{p!}$$

$$\sum_{p=2}^{\infty} \frac{1}{p!} = (n+1)! \frac{(e-2)}{\sum_{i=1}^{\infty} i!}$$

$$\Rightarrow (B_{n+1}) \leq (n+1)!$$

4) then $k \in \mathbb{N}$, $\frac{|B_n|}{n} \leq 1$.

$\frac{|B_n|}{n!} \leq 1$.
 $\frac{|B_n|}{n!} \leq 1 \Rightarrow \text{radius } \leq 1$

$\sum_{n=0}^{\infty} \frac{B_n}{n!} z^n = e^z$

$$\text{ona } (e^z - 1) \left(\sum_{n=0}^{\infty} \frac{B_n}{n!} z^n \right) = \left(\sum_{n=1}^{\infty} \frac{z^n}{n!} \right) \left(\sum_{n=0}^{\infty} \frac{B_n z^n}{n!} \right)$$

$$= \sum_{n=0}^{\infty} C_n z^n \text{ avec } -$$

$$\underline{a_0=0} \quad C_n = \sum_{k=0}^n a_k b_{n-k}$$

$$C_1 = a_0 b_1 + a_1 b_0$$

$$C_0 = 0$$

$$C_1 = B_0 = 1$$

$$\forall n \geq 2; C_n = \sum_{k=1}^n \frac{1}{k!} \cdot \frac{B_{n-k}}{(n-k)!} = \sum_{k=0}^{n-1} \frac{B_k}{k!} \frac{1}{(n-k)!} = 0$$

$$\Rightarrow (e^z - 1) \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n = z$$

$$\text{a) } \forall z \neq 0,$$

$$\text{b) } x \in \mathbb{R}, \mathbb{Z}$$

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n$$

$$i + \frac{e^{i\pi x}}{e^{2i\pi x} - 1} = i \frac{(e^{2i\pi x} + 1)}{e^{2i\pi x} - 1} = i \frac{(e^{i\pi x} + e^{-i\pi x})}{(e^{i\pi x} - e^{-i\pi x})}$$

$$= \cot \pi x$$

$$7) x \in \mathbb{R} \setminus \mathbb{Z}; |x| < \frac{R}{2\pi}$$

$$\pi x \cot(\pi x) = i\pi x + \frac{e^{i\pi x}}{e^{i\pi x} - 1}$$

$$\underline{5)} \quad i\pi x + \sum_{n=0}^{\infty} \frac{B_n}{n!} (e^{i\pi x})^n$$

$$= 1 + \sum_{n=2}^{\infty} \frac{B_n}{n!} (e^{i\pi x})^n$$

$$10) \text{ d'après } \pi x \cot(\pi x) = 1 + \sum_{n=2}^{\infty} \frac{B_n}{n!} (e^{i\pi x})^n$$

$$\text{et } \textcircled{5} \quad \quad \quad = 1 - 2 \sum_{n=1}^{\infty} \chi(n) x^{2n}$$

Par unicité des DSE (4)

$$B_{2n} = 0$$

$$\frac{B_{2n} (e^{i\pi x})^{2n}}{(2n)!} = -2\chi(n)$$

$$\Rightarrow \chi(n) = \frac{(-1)^{n+1} (2\pi)^{2n}}{2(2n)!} B_{2n}$$

$$ii) \quad \sum_{n \geq 0} \frac{B_{2n} 2^{2n}}{n!} = \sum_{n \geq 0} \frac{B_{2n} 2^{2n}}{(2n)!}$$

$$\left| \frac{W_{n+1}(3)}{W_n(3)} \right| = \left| \frac{B_{2n+2} \cdot |3|^{2n+2}}{B_{2n} \cdot (n+2)(2n+1)} \right|$$

$$= \left| \frac{\chi(n+2)}{(2\pi)^{2n+2}} \cdot \frac{x(e^{i\pi x})^{2n+2}}{x(e^{i\pi x})^{2n}} \cdot \frac{\chi(n)}{x(e^{i\pi x})^{2n}} \right|$$

$$\sim \frac{|3|^{2n}}{4\pi^2} \rightarrow \frac{|3|^{2n}}{4\pi^2}$$

$$\Rightarrow R = e\pi$$