

Problème Moment d'un v.a.d.

$$X(\omega) = \{x_1, \dots, x_r\}$$

$$\forall t \in \mathbb{R}^*; \varphi_X(t) = \frac{1}{t} \ln(\pi_X(t))$$

$$\pi_X(t) = E(e^{tx})$$

1) on a. $\pi_X(t) = \sum_{k=1}^n e^{tx_k} \frac{P(X=x_k)}{P_k}$
 $t \mapsto e^{tx_k}$ est ∞ sur $\mathbb{R} \rightarrow$
 $t \mapsto \pi_X(t)$ " " " " " " " " " " " "

$\forall n \in \mathbb{N}; \pi_X^{(n)}(t) = \sum_{k=1}^n x_k^n e^{tx_k} P_k$
 $\Rightarrow \pi_X^{(n)}(0) = \sum_{k=1}^n x_k^n P_k = E(X^n)$

2) a) on a $\pi_X(t) \geq 0, \forall t \in \mathbb{R}^*$
 on a $\sum_{k=1}^n P_k = 1 \Rightarrow \exists k \text{ tel que } P_k > 0$

$$\Rightarrow \sum_{k=1}^n e^{tx_k} P_k > 0 \quad \left| \begin{array}{l} \pi_X(t) = E(e^{tx}) \\ = 1 \end{array} \right.$$

$$\Rightarrow \pi_X(t) > 0, \forall t \in \mathbb{R}^*$$

$$\Rightarrow \varphi_X \text{ est bien définie sur } \mathbb{R}^*$$

$$\lim_{t \rightarrow 0} \varphi_X(t) = \lim_{t \rightarrow 0} \frac{\ln \pi_X(t) - \ln \pi_X(0)}{t - 0} = \left(\ln \pi_X \right)'(0)$$

$$\text{or } \ln \pi_X \text{ est dérivable sur } \mathbb{R}.$$

$$\text{et } \left(\ln \pi_X \right)'(0) = \frac{\pi_X'(0)}{\pi_X(0)} = \frac{E(X)}{E(1)} = E(X)$$

$$\Rightarrow \lim_{t \rightarrow 0} \varphi_X(t) = E(X)$$

b) on a $\frac{\varphi_X(t) - \varphi_X(0)}{t} = \frac{\ln \pi_X(t) - E(X)}{t}$
 $= \frac{\ln \pi_X(t) - t E(X)}{t^2}$

$$\text{ona. } \pi_X(t) = \sum_{k=1}^n e^{\frac{t x_k}{p_k}} = \sum_{k=1}^n p_k \left(1 + t x_k + \frac{t^2}{2} x_k^2 \right) + o(t^2)$$

$$= 1 + t E(X) + \frac{t^2}{2} E(X^2) + o(t^2)$$

$$\Rightarrow \ln(\pi_X(t)) = t E(X) + \frac{t^2}{2} E(X^2) - \frac{1}{2} \left(t E(X) + \frac{t^2}{2} E(X^2) \right)^2 + o(t^2)$$

$$= t E(X) + \frac{t^2}{2} (E(X^2) - (E(X))^2) + o(t^2)$$

$$= t E(X) + \frac{t^2}{2} V(X) + o(t^2)$$

$$\Rightarrow \varphi_X(t) = E(X) + \frac{t}{2} V(X) + o(t) \quad \text{au } 0$$

$$= \varphi_X''(0)$$

$$\Rightarrow \varphi_X \text{ est dérivable en } 0 \text{ et } \varphi_X'(0) = \frac{V(X)}{2}$$

di) soit $u \geq 0$, mg. $e^u \leq 1 + u + \frac{u^2}{2}$.

methode

$$f(u) = e^u \text{ est } C^2 \text{ sur } \mathbb{R}$$

D'après la formule de Taylor-Lagrange

$$\forall u < 0 \Rightarrow \exists c \in]u, 0[$$

$$f(u) = 1 + u + \frac{u^2}{2} + \frac{u^3}{3!} \cdot f^{(3)}(c).$$

$$e^u - \left(1 + u + \frac{u^2}{2}\right) = \frac{u^3}{6} e^c \leq 0$$

Autrement

$$e^u = \sum_{n=0}^{\infty} \frac{u^n}{n!} = 1 + u + \frac{u^2}{2} + \sum_{n=3}^{\infty} \frac{u^n}{n!}$$

$$\forall u < 0 \Rightarrow \sum_{n \geq 3} \frac{u^n}{n!} \text{ est atteinte}$$

$$\frac{u^n}{n!} \xrightarrow{n \rightarrow \infty} 0. \left(\text{car } \sum_{n \geq 0} \frac{u^n}{n!} < \infty \right)$$

$$\left| \frac{\frac{u^{n+1}}{(n+1)!}}{\frac{u^n}{n!}} \right| = \frac{|u|}{n+1} \xrightarrow{n \rightarrow \infty} 0$$

$$\Rightarrow \sum_{n \geq 3} \frac{|u|^n}{n!} \text{ converge } \Rightarrow \exists n_0 \in \mathbb{N} \forall n \geq n_0 \frac{|u|^n}{n!} < \frac{1}{2^n}$$

$$\Rightarrow \sum_{n \geq 3} \frac{|u|^n}{n!} < \sum_{n \geq 3} \frac{1}{2^n} < \frac{1}{2} < \frac{1}{2}$$

ii) Puisque $X(\Omega) \subset \mathbb{R}$.

d'après i) $e^{tX} \leq 1 + tX + \frac{t^2 X^2}{2}$

$$\Rightarrow E(e^{tX}) \leq 1 + tE(X) + \frac{t^2}{2} E(X^2)$$

$$\Rightarrow \ln(E(e^{tX})) \leq \ln\left(1 + tE(X) + \frac{t^2}{2} E(X^2)\right)$$

$$E(e^{tX}) > 0 \Rightarrow 1 + tE(X) + \frac{t^2}{2} E(X^2) > 0$$

$$\Rightarrow \ln(E(e^{tX})) \leq tE(X) + \frac{t^2}{2} E(X^2)$$

$$\Rightarrow \phi_X(t) \leq E(X) + \frac{t}{2} E(X^2)$$

d) i) $p_i(t) = e^{tx_i}$

$\Phi: f \mapsto f'$

ona. $\Phi(p_i)(t) = x_i \cdot e^{tx_i}$

$$\Rightarrow \Phi(p_i) = x_i \cdot p_i$$

Donc $f_i \in V_P$ do Φ associée à x_i .
 c) $(f_1, f_2, \dots, f_r)$ est une famille de V_P
 do Φ associée à des V_P $x_i \neq$.
 Alors elle est libre.

Autrement :

Soit $\alpha_1, \dots, \alpha_r \in \mathbb{R}$ tq.
 $\sum_{k=1}^r \alpha_k f_k = \vec{0} \Leftrightarrow \sum_{k=1}^r \alpha_k e^{x_k} = 0, \forall x \in \mathbb{R}$.

$\sum_{k=1}^r \alpha_k = 0$
 $\sum_{k=1}^r \alpha_k x_k = 0 \Rightarrow \sum_{k=1}^r \alpha_k x_k^2 = 0$
 $\vdots$
 $\sum_{k=1}^r \alpha_k x_k^{n-1} = 0$

Alors.

$$\begin{cases} \alpha_1 + \alpha_2 + \dots + \alpha_r = 0 \\ \alpha_1 x_1 + \dots + \alpha_r x_r = 0 \\ \vdots \\ \alpha_1 x_1^{n-1} + \dots + \alpha_r x_r^{n-1} = 0 \end{cases}$$

$$\Leftrightarrow \begin{pmatrix} 1 & \dots & 1 \\ x_1 & \dots & x_r \\ \vdots & & \vdots \\ x_1^{n-1} & \dots & x_r^{n-1} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_r \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

(b) $A = \text{universelle}$
 $\det A = V(x_1, \dots, x_r) \neq 0$ car $x_i \neq x_j$.

$$\Leftrightarrow \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_r \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

(ii) Si X et Y ont la m. loi

$$\Rightarrow E(e^{tx}) = E(e^{ty})$$

$$\Rightarrow \varphi_X(t) = \varphi_Y(t), \forall t \neq 0$$

$$\text{et } \varphi_X(0) = E(X) = E(Y) = \varphi_Y(0)$$

$$\Rightarrow \varphi_X = \varphi_Y.$$

" $\Leftarrow$ " On a. $\varphi_X(t) = \varphi_Y(t), \forall t \neq 0$

$$\Rightarrow E(e^{tx}) = E(e^{ty})$$

$$\Leftrightarrow \sum_{k=1}^n p_k e^{tx_k} = \sum_{k=1}^n q_k e^{tx_k} \quad \left(\begin{array}{l} X(0) = Y(0) \\ q_k = P(Y=x_k) \end{array} \right)$$

$$\sum_{k=1}^n (p_k - q_k) e^{tx_k} = 0$$

$$\Rightarrow \forall k \in \{1, \dots, n\}, p_k - q_k = 0. \text{ car } (e^{tx_k})_{k=1}^n \text{ libre}$$

$$\Rightarrow p_k = q_k$$

$\Rightarrow X$ et Y ont la m. loi.

$$\text{d) } \varphi_{X_1 + \dots + X_n}(t) = \frac{1}{t} \ln(\pi_{X_1 + \dots + X_n}(t)) \quad \text{Si } t \neq 0$$

$$= \frac{1}{t} \ln\left(\prod_{i=1}^n \pi_{X_i}(t)\right)$$

$$= \frac{1}{t} \sum_{i=1}^n \ln(\pi_{X_i}(t)) = \sum_{i=1}^n \varphi_{X_i}(t)$$

$$\text{Si } t=0, \varphi_{X_1 + \dots + X_n}(0) = E(X_1 + \dots + X_n)$$

$$= \sum_{i=1}^n E(X_i)$$

$$= \sum_{i=1}^n \varphi_{X_i}(0)$$

d) $X \sim B(n, p)$.

$$\pi_X(t) = E(e^{tx}) = \sum_{k=0}^n e^{tk} \cdot p(x=k)$$

$$= \sum_{k=0}^n e^{tk} \cdot \binom{n}{k} (1-p)^{n-k} p^k$$

$$= \sum_{k=0}^n \binom{n}{k} (p e^{t(1-p)})^k (1-p)^{n-k}$$

$$= (p e^{t(1-p)} + 1-p)^n$$

g) 17. X est symétrique $\Leftrightarrow \varphi_X$ est impaire.

" $\Rightarrow$ " " X et $-X$ ont la même loi $\stackrel{\text{dii)}}{\Rightarrow} \varphi_{-X} = \overline{\varphi_X}$.

$$a) \varphi_X(-t) = \overline{\varphi_X(t)}$$

$$= -\frac{1}{t} \ln \left(E(e^{-X(t)}) \right) = -\varphi_X(t) \quad \forall t \neq 0.$$

$$\varphi_X(0) = E(X) = E(-X) = -E(X) = -\varphi(0).$$

" $\Leftarrow$ " On a φ_X est impaire $\Rightarrow \varphi_X = \overline{\varphi_X} \Rightarrow X$ et $-X$ ont la même loi.

$\Rightarrow (X_n)_{n \geq 1}$ sont mut. indép., $\mu = E(X_n)$, $\forall n \in \mathbb{N}^*$.

$$\sigma = \sqrt{V(X_n)}$$

$$S_n = X_1 + \dots + X_n; \quad S_n^* = \frac{S_n - E(S_n)}{\sqrt{V(S_n)}}$$

$$a) \varphi_{S_n^*}(t) = \frac{1}{t} \ln \left(E(e^{t S_n^*}) \right) \quad \Rightarrow \quad \begin{aligned} E(S_n) &= n \cdot \mu \\ V(S_n) &= n \cdot \sigma^2 \end{aligned}$$

$$\forall t \neq 0, E(e^{t S_n^*}) = E\left(e^{\frac{t(S_n - E(S_n))}{\sigma \sqrt{n}}}\right)$$

$$\Rightarrow E(e^{tS_n^*}) = e^{-\frac{t \sqrt{n} \cdot m}{\sigma}} E(e^{\frac{tS_n}{\sigma \sqrt{n}}})$$

$$\Rightarrow \frac{\ln(E(e^{tS_n^*}))}{t} = -\frac{\sqrt{n} \cdot m}{\sigma} + \frac{\ln(E(e^{\frac{tS_n}{\sigma \sqrt{n}}})}{t}$$

$\Rightarrow \forall t \neq 0$

$$\varphi_{S_n^*}(t) = -\frac{\sqrt{n} \cdot m}{\sigma} + \frac{\ln E(e^{\frac{tS_n}{\sigma \sqrt{n}}})}{t}$$

$$= -\frac{\sqrt{n} \cdot m}{\sigma} + \frac{1}{\frac{t}{\sigma \sqrt{n}}} \cdot \frac{\ln(E(e^{\frac{tS_n}{\sigma \sqrt{n}}})}{\frac{t}{\sigma \sqrt{n}}}$$

$$= -\frac{\sqrt{n} \cdot m}{\sigma} + \frac{1}{\frac{t}{\sigma \sqrt{n}}} \varphi_n\left(\frac{t}{\sigma \sqrt{n}}\right)$$

$$\varphi_{S_n^*}(t) = -\frac{\sqrt{n} \cdot m}{\sigma} + \frac{1}{\frac{t}{\sigma \sqrt{n}}} \varphi_n\left(\frac{t}{\sigma \sqrt{n}}\right)$$

Wg h. $\varphi_{S_n^*}(t) = \frac{1}{2}$
n.H.

$$t \in \mathbb{R}^*: n \rightarrow \infty \Rightarrow \frac{t}{\sigma \sqrt{n}} \rightarrow 0$$

$$\text{ona. } \varphi_x(t) = E(X) + \frac{t}{2} V(X) + o(t)$$

$$\Rightarrow \varphi_x\left(\frac{t}{\sigma \sqrt{n}}\right) = \widehat{E(X)} + \frac{t}{2\sigma \sqrt{n}} V(X) + o\left(\frac{1}{\sigma \sqrt{n}}\right)$$

$$\Rightarrow \frac{\sqrt{n}}{\sigma} \varphi_x\left(\frac{t}{\sigma \sqrt{n}}\right) = \frac{m \sqrt{n}}{\sigma} + \frac{t}{2} V(X) + o(1)$$

$$\Rightarrow \varphi_{S_n^*}(t) = \frac{1}{2}$$

Partie 1.

$$\begin{aligned} & \text{1) } 0 \leq X \leq 1 \text{ h. } \varphi_X(t) = \frac{1}{2} \\ & \text{2) } 0 \leq X \leq 1 \text{ h. } \varphi_X(t) = \frac{1}{2} \\ & \text{3) } 0 \leq X \leq 1 \text{ h. } \varphi_X(t) = \frac{1}{2} \end{aligned}$$

Probleme: "Autovermindernde Poisson"

$$5) I_n = \{k \in \mathbb{N} / n d + a \sqrt{n} \leq k \leq n d + b \sqrt{n}\}$$

$$6) a) P(a \leq T_n \leq b) = P\left(a \leq \frac{S_n - n d}{\sqrt{n d}} \leq b\right) \\ = P(a \sqrt{n d} + n d \leq S_n \leq b \sqrt{n d} + n d)$$

$$= P\left(\bigcup_{k \in I_n} (S_n = k)\right) = \sum_{k \in I_n} P(S_n = k)$$

$$b) \lim_{n \rightarrow \infty} P(a \leq T_n \leq b) = \frac{1}{\sqrt{2\pi}} \int_a^b f(t) dt$$

→ Poisson-Leibniz

$$7) \text{ Dann } \int_0^{\infty} f(t) dt \leq C \Rightarrow \lim_{C \rightarrow \infty} \frac{1}{\sqrt{2\pi}} \int_C^{\infty} f(t) dt = 0 \\ \forall \epsilon > 0, \exists C_1 > 0 \text{ s.d. } \forall C > C_1, \frac{1}{\sqrt{2\pi}} \int_C^{\infty} f(t) dt < \epsilon$$

$$8) \text{ Seien } \epsilon > 0 \text{ et } c > \max(c_1, c_2)$$

$$\left| P(T_n \geq a) - \frac{1}{\sqrt{2\pi}} \int_a^{\infty} f(t) dt \right| = \\ \left| P(a \leq T_n \leq c) + P(T_n > c) - \frac{1}{\sqrt{2\pi}} \int_a^c f(t) dt - \frac{1}{\sqrt{2\pi}} \int_c^{\infty} f(t) dt \right|$$

$$\leq \left| P(a \leq T_n \leq c) - \frac{1}{\sqrt{2\pi}} \int_a^c f(t) dt \right| + P(T_n > c) + \frac{1}{\sqrt{2\pi}} \int_c^{\infty} f(t) dt$$

≤ ε

$$2. (T_n > c) \subset (|T_n| > c) \\ \Rightarrow P(T_n > c) \leq P(|T_n| > c) \cdot \epsilon$$

$$\Rightarrow \forall \epsilon > 0, \exists n_0 \in \mathbb{N} \text{ s.d. } \forall n \geq n_0 \\ |P(T_n \geq a) - \frac{1}{\sqrt{2\pi}} \int_a^{\infty} f(t) dt| < \epsilon$$

$$1) a) (T_n = a) \subset (a \leq T_n \leq a + \epsilon)$$

$$\Rightarrow P(T_n = a) \leq P(a \leq T_n \leq a + \epsilon), \forall \epsilon > 0$$

$$\Rightarrow \lim_{n \rightarrow \infty} P(T_n = a) \leq \lim_{n \rightarrow \infty} P(a \leq T_n \leq a + \epsilon) = \frac{1}{\sqrt{2\pi}} \int_a^{a+\epsilon} f(t) dt, \forall \epsilon > 0$$

$$\lim_{\epsilon \rightarrow 0} F(\epsilon) = F(0) = 0$$

$\epsilon \rightarrow F(\epsilon)$ est continue

$$\Rightarrow \lim_{n \rightarrow \infty} P(T_n = a) = 0$$

$$ii) A_n = \sum_{k=0}^{E(nd)} \frac{(nd)^k}{k!}$$

$$\begin{aligned} \text{or a. } P(T_n \leq 0) &= P(S_n \leq nd) = P\left(\bigcup_{k \leq nd} S_n = k\right) = P\left(\bigcup_{k=0}^{E(nd)} (S_n = k)\right) \\ &= \sum_{k=0}^{E(nd)} P(S_n = k) = \sum_{k=0}^{E(nd)} \frac{(nd)^k}{k!} = A_n \end{aligned}$$

$$\begin{aligned}
 b) \quad \lim_{n \rightarrow \infty} \mathbb{E}^{nd} A_n &= \lim_{n \rightarrow \infty} P(T_n \leq 0) \\
 &= 1 - \lim_{n \rightarrow \infty} P(T_n > 0) \\
 &= 1 - \lim_{n \rightarrow \infty} P(T_n \geq 0) \\
 &= 1 - \frac{1}{\sqrt{2\pi}} \int_0^{\infty} f(t) dt = \frac{1}{2}
 \end{aligned}$$

$$\begin{aligned}
 & \text{8) soit } \varepsilon > 0 \text{ et } C > \max(C_1, C_2) \\
 & \left| P(T_n \geq a) - \frac{1}{\sqrt{2\pi}} \int_a^{\infty} f(t) dt \right| = \\
 & \left| P(a \leq T_n \leq C) + P(T_n > C) - \frac{1}{\sqrt{2\pi}} \int_a^C f(t) dt - \frac{1}{\sqrt{2\pi}} \int_C^{\infty} f(t) dt \right|
 \end{aligned}$$

$$\leq \underbrace{\left| P(a \leq T_n \leq C) - \frac{1}{\sqrt{2\pi}} \int_a^C f(t) dt \right|}_{\leq \varepsilon} + \underbrace{P(T_n > C) + \frac{1}{\sqrt{2\pi}} \int_C^{\infty} f(t) dt}_{\leq \varepsilon}$$

$$\begin{aligned}
 & \text{or } (T_n > C) \subset (|T_n| > C) \\
 & \Rightarrow P(T_n > C) \leq P(|T_n| > C) \cdot \sqrt{2} \\
 & \Rightarrow \forall \varepsilon > 0, \exists n_0 \in \mathbb{N} \text{ s.t. } n \geq n_0
 \end{aligned}$$

$$\left| P(T_n \geq a) - \frac{1}{\sqrt{2\pi}} \int_a^{\infty} f(t) dt \right| \leq \frac{1}{\sqrt{2\pi}} \int_a^{\infty} f(t) dt + \frac{1}{\sqrt{2\pi}} \int_C^{\infty} f(t) dt$$