

Exercice : 1

Résoudre les équations différentielles suivantes d'inconnue $y : I \rightarrow \mathbb{R}$:

1) $(1+x^2)y' + xy = 1$, $I = \mathbb{R}$

2) $2xy' + y = \frac{1}{1+x}$, $I =]0, +\infty[$

3) $y'' + 6y' + 9y = \frac{e^{-3x}}{\sqrt{1+x^2}}$, $I = \mathbb{R}$

4) $y'' - 2y' + y = sh(x)$, $I = \mathbb{R}$

5) $x^2y'' - 4xy' + 6y = 0$, $I =]0, +\infty[$

(indication : chercher des solution polynomiales)

Exercice : 2 Résoudre les systèmes différentielles suivantes :

1) $\begin{cases} x' = y + t^2 \\ y' = x - t^2 \end{cases}$, 2) $\begin{cases} x' = x + 2y - z \\ y' = 2x + 4y - 2z \\ z' = -x - 2y + z \end{cases}$, 3) $\begin{cases} x' = 3x + y - z + 1 \\ y' = x + y + z + e^t \\ z' = 2x + 2z \end{cases}$ 4) $\begin{cases} x' = x - y \\ y' = x + y \end{cases}$

Exercice : 3

5) $\begin{cases} x' = -y + 2z \\ y' = -5x + 4y - 4z \\ z' = -3x + 3y - 4z \end{cases}$
 Résoudre les équations différentielles suivantes d'inconnue $y : I \rightarrow \mathbb{R}$:

1) $y'' + y = \frac{1}{\sin(x)}$ sur $I =]0, \pi[$.

2) $(2x+1)y'' + (4x-2)y' - 8y = 0$, $I =]-\frac{1}{2}, +\infty[$

(Indication : montrer qu'elle admet une solution de la forme $e^{\alpha x}$)

3) $x^2y'' + xy' + y = \sin(\alpha \log(x))$, $I =]0, +\infty[$

(Indication : utiliser changement de variable $t = \log(x)$)

Exercice : 4

On considère l'équation différentielle (E) : $xy'' + y' + xy = 0$

1) Que peut-on dire de l'ensemble des solution de (E).

2) Trouver les solutions de (E) développables en séries entière , préciser la solution J telle que $J(0) = 1$.

Ex 2:

$$\begin{cases} x' = y + t^2 \\ y' = x - t^2 \end{cases}$$

$$x'(t) = A(t)x(t) + B(t)$$

$$\text{avec } x(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$B(t) = \begin{pmatrix} t^2 \\ -t^2 \end{pmatrix}$$

$$\chi_A(x) = x^2 - 1 = (x-1)(x+1)$$

$\Rightarrow A$ diagonalisable.

$$v_1 \in E_1(A) \Leftrightarrow \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Leftrightarrow -x_1 + x_2 = 0 \Leftrightarrow x_1 = x_2$$

$$\Rightarrow E_1(A) = \text{vect} \left\{ \underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{v_1} \right\}$$

$$v_2 \in E_{-1}(A) \Leftrightarrow \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Leftrightarrow x_1 + x_2 = 0 \Leftrightarrow x_1 = -x_2$$

$$\Rightarrow E_{-1}(A) = \text{vect} \left\{ \underbrace{\begin{pmatrix} -1 \\ 1 \end{pmatrix}}_{v_2} \right\}$$

$$(e^t v_1, e^{-t} v_2) : \text{S.F.S de } (E_0)$$

tt sol^e de (E_0) s'écrit: $\alpha, \beta \in \mathbb{K}$

$$x_0(t) = \alpha e^t v_1 + \beta e^{-t} v_2$$

soit $x_p(t)$ une sol^e particulière de (E) :

$$x_p(t) = \alpha(t) e^t v_1 + \beta(t) e^{-t} v_2$$

$$\Rightarrow x'_p(t) = \alpha' e^t v_1 + \alpha e^t v_1 + \beta' e^{-t} v_2 - \beta e^{-t} v_2$$

$$\text{or } x'_p(t) = A x_p(t) + B(t)$$

on injecte:

$$\Rightarrow \alpha' e^t v_1 + \beta' e^{-t} v_2 = B(t) \quad \text{on obtient cette forme}$$

$$\alpha' e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \beta' e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} t^2 \\ -t^2 \end{pmatrix}$$

$$\Rightarrow \begin{cases} \alpha' e^t + \beta' e^{-t} = t^2 \\ \alpha' e^t - \beta' e^{-t} = -t^2 \end{cases}$$

$$2\alpha' e^t = 0 \Rightarrow \alpha' = 0 \Rightarrow \alpha = 0$$

$$2\beta' e^{-t} = -t^2 \Rightarrow \beta' = -\frac{t^2}{2} e^t$$

$$\int \frac{t^2}{u(t) v'(t)} dt = \left[t^2 e^t \right]_0^x - \int_0^x t^2 e^t dt$$

$$= x^2 e^x - 2 \left[[t e^t]_0^x - \int_0^x e^t dt \right]$$

$$= x^2 e^x - 2x e^x + 2e^x$$

$$= e^x (x^2 - 2x + 2)$$

$$x_p(t) = (t^2 - 2t + 2) v_2$$

tt sol^e de (E) s'écrit sous la forme:

$$x(t) = \alpha e^t v_1 + \beta e^{-t} v_2 + (t^2 - 2t + 2) v_2$$

$$\begin{cases} x(t) = \alpha e^t - \beta e^{-t} - (t^2 + 2t + 2) \\ y(t) = \alpha e^t + \beta e^{-t} + (t^2 + 2t + 2) \end{cases}$$

ou bien = 2^e méthode:

$$P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, A = P D P^{-1}$$

$$A x = x'$$

$$\Leftrightarrow P D P^{-1} x = x'$$

$$\Leftrightarrow D P^{-1} x = P^{-1} x'$$

$$\text{on pose } y(t) = P^{-1} x(t)$$

$$(S) \Leftrightarrow D y(t) = y'(t)$$

$$D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$$

$$\Leftrightarrow \begin{cases} y_1(t) = y_1'(t) \\ -y_2(t) = y_2'(t) \end{cases}$$

$$\Leftrightarrow \begin{cases} y_1(t) = \alpha e^t \\ y_2(t) = -\alpha e^{-t} \end{cases}$$

$$\Rightarrow x(t) = p y(t) = \begin{pmatrix} \alpha e^t + \beta e^{-t} \\ \alpha e^t - \beta e^{-t} \end{pmatrix}$$

$$\Rightarrow \begin{cases} x(t) = \alpha e^t + \beta e^{-t} \\ y(t) = \alpha e^t - \beta e^{-t} \end{cases}$$

$$\textcircled{S}: \begin{cases} x' = x + 2y - z \\ y' = 2x + 4y - 2z \\ z' = -x - 2y + z \end{cases}$$

$$\textcircled{S} \quad \textcircled{E} X'(t) = A X(t)$$

$$\text{avec: } X(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \\ -1 & -2 & 1 \end{pmatrix}$$

$$\chi_A(x) = \begin{vmatrix} x-1 & -2 & 1 \\ -2 & x-4 & 2 \\ 1 & 2 & x-1 \end{vmatrix}$$

$$\begin{aligned} & l_3 \leftarrow l_1 + l_3 \\ & = \begin{vmatrix} x-1 & -2 & 1 \\ -2 & x-4 & 2 \\ x & 0 & x \end{vmatrix} \end{aligned}$$

$$\begin{aligned} & c_3 \leftarrow c_1 - c_3 \\ & = \begin{vmatrix} x-1 & -2 & -x+2 \\ -2 & x-4 & +4 \\ x & 0 & 0 \end{vmatrix} \end{aligned}$$

$$= x \begin{vmatrix} -2 & -x+2 \\ x-4 & +4 \end{vmatrix} = x(x^2 - 6x) = x^2(x-6)$$

$$E_0(A) = ?$$

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in E_0(A) \quad \textcircled{E} : AX = 0$$

$$\textcircled{E} \begin{cases} x + 2y - z = 0 \\ 2x + 4y - 2z = 0 \\ -x - 2y + z = 0 \end{cases}$$

$$\textcircled{E} \quad x + 2y - z = 0 \quad \textcircled{E} \quad x = -2y + z$$

$$E_0(A) = \text{vect} \left\{ \underbrace{\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}}_{v_1}, \underbrace{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}}_{v_2} \right\}$$

$$\Rightarrow A \text{ est diag} \quad (\dim(E_0(A)) = m_0 = 2)$$

$$E_6(A) = ?$$

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in E_6(A) \quad \textcircled{E} : AX = 6X$$

$$\textcircled{E} \begin{cases} x + 2y - z = 6x \\ 2x + 4y - 2z = 6y \\ -x - 2y + z = 6z \end{cases}$$

$$\textcircled{E} \begin{cases} x = -z \\ y = 2x \end{cases}$$

$$E_6(A) = \text{vect} \left\{ \underbrace{\begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}}_{v_3} \right\}$$

$$\Rightarrow (v_1, v_2, e^{6t} v_3) \text{ S.F.S de } \textcircled{S}.$$

$$\text{tt sol}^\circ \text{ de } \textcircled{S} \text{ est de la forme:}$$

$$X(t) = \alpha v_1 + \beta v_2 + \gamma v_3 e^{6t}$$

$$\begin{cases} x(t) = -2\alpha + \beta - \gamma e^{6t} \\ y(t) = \alpha - 2\gamma e^{6t} \\ z(t) = \beta + \gamma e^{6t} \end{cases}$$

$$\times \text{2ème méthode:}$$

$$DY = Y' \quad P = \begin{pmatrix} -2 & 1 & -1 \\ 1 & 0 & -2 \\ 0 & 1 & 1 \end{pmatrix} \text{ avec: } Y = P^{-1} X$$

$$P = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 6 \end{pmatrix}$$

$$\begin{cases} y_1' = 0 \\ y_2' = 0 \\ y_3' = 6y_3 \end{cases} \quad (\Leftrightarrow) \quad \begin{cases} y_1 = \gamma \\ y_2 = \beta \\ y_3 = \alpha e^{6t} \end{cases}$$

$$x(t) = P Y(t)$$

$$= \begin{pmatrix} -2\gamma + \beta - \alpha e^{6t} \\ \gamma - 2\alpha e^{6t} \\ \beta + \alpha e^{6t} \end{pmatrix}$$

$$3)(E): \begin{cases} x' = 3x + y - z + 1 \\ y' = x + y + z + e^t \\ z' = 2x + 2y \end{cases}$$

$$(E) \Leftrightarrow x'(t) = Ax + B(t)$$

$$\text{avec: } x(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}; \quad A = \begin{pmatrix} 3 & 1 & -1 \\ 1 & 1 & 1 \\ 2 & 0 & 2 \end{pmatrix}$$

$$B(t) = \begin{pmatrix} 1 \\ e^t \\ 0 \end{pmatrix}$$

$$\chi_A(x) = \begin{vmatrix} x-3 & -1 & 1 \\ -1 & x-1 & -1 \\ -2 & 0 & x-2 \end{vmatrix}$$

$$c_3 \leftarrow c_3 + c_2$$

$$= \begin{vmatrix} x-3 & -1 & 0 \\ -1 & x-1 & x-2 \\ -2 & 0 & x-2 \end{vmatrix}$$

$$= L_2 \leftarrow L_2 - L_3$$

$$= \begin{vmatrix} x-3 & -1 & 0 \\ 1 & x-1 & 0 \\ -2 & 0 & x-2 \end{vmatrix}$$

$$= (x-2) \begin{vmatrix} x-3 & -1 \\ 1 & x-1 \end{vmatrix} = (x-2)^3$$

$$\text{on pose: } N = A - 2I_3 \Rightarrow A = N + 2I_3$$

$$N^3 = 0$$

$$e^{tA} = e^{t(N+2I_3)} = e^{tN} e^{2tI_3}$$

$$N = A - 2I_3 = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 2 & 0 & 0 \end{pmatrix}$$

$$N^2 = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 2 & -2 \\ 2 & 2 & -2 \end{pmatrix}$$

$$e^{tN} = \sum_{k=0}^{+\infty} \frac{1}{k!} t^k N^k$$

$$= I_3 + tN + \frac{1}{2} t^2 N^2$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} t & t & -t \\ t & -t & t \\ 2t & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ t^2 & t^2 & -t^2 \\ t^2 & t^2 & -t^2 \end{pmatrix}$$

$$= \begin{pmatrix} 1+t & t & -t \\ t+t^2 & 1-t+t^2 & 1-t^2 \\ 2t+t^2 & t^2 & 1-t^2 \end{pmatrix}$$

$$e^{tA} = e^{2t} e^{tN}$$

$$\text{toute sol: } x(t) = e^{tA} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} / (\alpha, \beta, \gamma) \in \mathbb{K}$$

$$\text{soient: } x_1(t) = e^{2t} \begin{pmatrix} 1+t \\ t+t^2 \\ 2t+t^2 \end{pmatrix}$$

$$x_2(t) = e^{2t} \begin{pmatrix} t \\ 1-t+t^2 \\ t^2 \end{pmatrix}; \quad x_3(t) = e^{2t} \begin{pmatrix} -t \\ 1-t^2 \\ 1-t^2 \end{pmatrix}$$

$$(x_1, x_2, x_3) \text{ s.f.s de } (E_0).$$

$$x_p(t) = \alpha(t) x_1(t) + \beta(t) x_2(t) + \gamma(t) x_3(t)$$

$$(E) \quad \alpha' x_1 + \beta' x_2 + \gamma' x_3 = B$$

③

cherchons α', γ', β' .

ou bien : $\chi_A(x) = (x-2)^3$

$$\text{rg}(A - 2I_3) = \text{rg} \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 2 & 0 & 0 \end{pmatrix} = 2$$

$$\dim(E_2(A)) = 1$$

on calcule $(A - 2I_3)^2$

on choisit $v_2 \in \ker(A - 2I_3) \setminus \ker(A - 2I_3)^2$

$$v_1 = (A - 2I_3)v_2$$

$$(A - 2I_3)v_1 = 0 \Rightarrow v_1 \in \ker(A - 2I_3)$$

$$Av_2 = v_1 + 2v_2$$

$$Av_1 = 2v_1$$

$$B = (v_1, v_2, v_3)$$

$$M_B(t) = \begin{pmatrix} f(v_1) & f(v_2) & f(v_3) \\ 2 & 1 & x \\ 0 & 2 & x \\ 0 & 0 & 2 \end{pmatrix} \begin{matrix} v_1 \\ v_2 \\ v_3 \end{matrix}$$

$= T$

v_3 est un vecteur de la base.

canonique libre avec (v_1, v_2) .

$$4) \begin{cases} x' = x - y \\ y' = x + y \end{cases}$$

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$\chi_A(x) = \begin{vmatrix} x-1 & +1 \\ -1 & x-1 \end{vmatrix} = (x-1)^2 + 1 = x^2 - 2x + 2$$

$$\Delta' = -1 < 0$$

* Résolution dans $\mathbb{C}$:

$$\begin{cases} x_1 = 1-i \\ x_2 = 1+i \end{cases}$$

$\Rightarrow A$ est diag dans $\mathbb{C}$.

$$E_{1+i}(A) = ? \quad Ax = (1+i)x$$

$$\Leftrightarrow \begin{cases} x - y = (1+i)x \\ x + y = (1+i)y \end{cases} \Leftrightarrow \begin{cases} y = -ix \\ x = iy \end{cases}$$

$$E_{1+i}(A) = \text{vect} \left\{ \begin{pmatrix} 1 \\ -i \end{pmatrix} \right\}$$

v_1

$$E_{1-i}(A) = \text{vect} \left\{ \begin{pmatrix} 1 \\ i \end{pmatrix} \right\}$$

v_2

$v_1 = \overline{v_2}$

$$\left(e^{(1+i)t} v_1, e^{(1-i)t} v_2 \right) \text{ S.F.S de } (E_0)$$

toute solⁿ dans $\mathbb{C}$ est de la forme :

$$x(t) = \alpha e^{(1+i)t} v_1 + \beta e^{(1-i)t} v_2$$

$$(\alpha, \beta) \in \mathbb{C}^2$$

* Résolution dans $\mathbb{R}$:

$$\text{on a : } v_2 = \overline{v_1}$$

$$\frac{1}{2}(x_1 + x_2) \text{ sont des solⁿ de } (E).$$

$$\frac{1}{2i}(x_1 - x_2)$$

$$\text{or } x_2 = \overline{x_1}$$

$$\frac{1}{2}(x_1 + x_2) = \text{Re} \left(e^{(1+i)t} \begin{pmatrix} 1 \\ -i \end{pmatrix} \right)$$

$$= \text{Re} \begin{pmatrix} e^{(1+i)t} \\ -i e^{(1+i)t} \end{pmatrix}$$

$$= \begin{pmatrix} e^t \cos(t) \\ e^t \sin(t) \end{pmatrix} y_1(t)$$

$$\frac{1}{2i}(x_1 - x_2) = \text{Im} \left(e^{(1+i)t} \begin{pmatrix} 1 \\ -i \end{pmatrix} \right) = \begin{pmatrix} e^t \sin(t) \\ -e^t \cos(t) \end{pmatrix} y_2(t)$$

* diag

* diag dans $\mathbb{C}$

* non diag : $\chi_A(x) = (x-\alpha)^3$

$$A = N + \alpha I_3 \Rightarrow e^{tA}$$

$$5/ \begin{cases} x' = -y + 2z \\ y' = -5x + 4y - 4z \\ z' = -3x + 3y - 4z \end{cases}$$

$$X(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}; A = \begin{pmatrix} 0 & -1 & 2 \\ -5 & 4 & -4 \\ -3 & 3 & -4 \end{pmatrix}$$

$$\chi_A(x) = \begin{vmatrix} x & 1 & -2 \\ 5 & x-4 & 4 \\ 3 & -3 & x+4 \end{vmatrix}$$

$$L_2 \leftarrow L_2 + C_1$$

$$= \begin{vmatrix} x & x+1 & -2 \\ 5 & x+1 & 4 \\ 3 & 0 & x+4 \end{vmatrix}$$

$$L_2 \leftarrow L_2 - L_1$$

$$= \begin{vmatrix} x & x+1 & -2 \\ 5-x & 0 & 6 \\ 3 & 0 & x+4 \end{vmatrix}$$

$$= x+1 \begin{vmatrix} 5-x & 6 \\ 3 & x+4 \end{vmatrix} = (x+1)^2 (x+2)$$

$$E_{-1}(A) = ? \quad (A + I_3)X = 0$$

$$\begin{pmatrix} 1 & -1 & 2 \\ -5 & 5 & -4 \\ -3 & 3 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$(E) \begin{cases} x - y + 2z = 0 \\ -5x + 5y - 4z = 0 \\ -3x + 3y - 3z = 0 \end{cases}$$

$$(E) \begin{cases} z = 0 \\ x = y \end{cases} \quad E_{-1}(A) = \text{vect} \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$$

(5)

$\Rightarrow A$ est trigonalisable.

$$\chi(A) = (x+1)^2 (x+2)$$

$$(A + 2I_3) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} +2 & -1 & 2 \\ -5 & 6 & -4 \\ -3 & 3 & -2 \end{pmatrix}$$

$$(E) \begin{cases} +2x - y + 2z = 0 \\ -5x + 6y - 4z = 0 \\ -3x + 3y - 2z = 0 \end{cases}$$

$$\otimes v_1 = \vec{v}_p \text{ associé à } -1$$

$$v_2 = \vec{v}_p \text{ associé à } -2$$

$$B = (v_1, v_2, v_3)$$

$$M_B \left(\begin{smallmatrix} f \\ T \end{smallmatrix} \right) = \begin{pmatrix} f(v_1) & f(v_2) & f(v_3) \\ -1 & 0 & ? \\ 0 & -2 & ? \\ 0 & 0 & -1 \end{pmatrix}$$

$$\text{on prend } v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{vmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = 1 \neq 0$$

$$P = \begin{pmatrix} v_1 & v_2 & v_3 \\ 2 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$A = PTP^{-1}$$

$$Av_3 = \begin{pmatrix} 2 \\ -4 \\ -4 \end{pmatrix} = \alpha v_1 + \beta v_2 - v_3$$

cherchons α et β ?

$$AX = X'$$

$$PTP^{-1}X = X'$$

$$TY = Y' \text{ avec } Y = P^{-1}X$$

$$\begin{cases} -2y_1 + \alpha y_3 = y'_1 \\ -y_2 + \beta y_3 = y'_2 \\ -y_3 = y'_3 \end{cases} \Leftrightarrow \begin{cases} y_3(t) = c e^{-t} \\ y'_2 = -y_2 + \beta c \\ y'_1 = -2y_1 + \alpha c e^{-t} \end{cases}$$