

Exercice 3 : Energie dissipée par effet Joule ds le cadre de l'effet de peau

1-a) $\text{div } \vec{E} = \frac{\rho}{\epsilon_0}$; $\text{div } \vec{B} = 0$; $\text{rot } \vec{E} = -\frac{\partial \vec{B}}{\partial t}$; $\text{rot } \vec{B} = \mu_0 \vec{j} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$
 or $\vec{j} = \gamma \vec{E} \rightarrow \text{rot } \vec{B} = \mu_0 \gamma \vec{E} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$

1-b) $\left\{ \begin{array}{l} \text{div } \vec{E} = \frac{\rho}{\epsilon_0} \\ \text{div } \vec{j} + \frac{\partial \rho}{\partial t} = 0 \\ \vec{j} = \gamma \vec{E} \end{array} \right\} \rightarrow \gamma \text{div } \vec{E} + \frac{\partial \rho}{\partial t} = 0 \quad \left| \quad \frac{\gamma}{\epsilon_0} \rho + \frac{\partial \rho}{\partial t} = 0 \right.$

1-c) *) $\frac{\gamma}{\epsilon_0} \rho + \frac{\partial \rho}{\partial t} = 0 \rightarrow \frac{\gamma}{\epsilon_0} \rho + i\omega \rho = \rho \left[\frac{\gamma}{\epsilon_0} + i\omega \right] = 0$
 $\neq 0$

$\rightarrow \rho = 0$

*) $\frac{j_0}{j} = \frac{\epsilon_0 \frac{\partial E}{\partial t}}{\gamma E} = \frac{\epsilon_0}{\gamma T} \ll 1 \rightarrow j_0 \ll j$

1-d) *) $\text{rot}(\text{rot } \vec{E}) = -\Delta \vec{E} + \text{grad}(\text{div } \vec{E}) = -\Delta \vec{E} = -\Delta \left(\frac{\vec{j}}{\gamma} \right)$
 $\text{rot} \left(-\frac{\partial \vec{B}}{\partial t} \right) = -\frac{\partial}{\partial t} (\text{rot } \vec{B}) = -\frac{\partial}{\partial t} (\mu_0 \vec{j})$

$\rightarrow \Delta \vec{j} - \mu_0 \gamma \frac{\partial \vec{j}}{\partial t} = \vec{0}$

*) on a $\vec{j} = j_0 e^{-3/8} e^{i(\omega t - 3/8)} \vec{u}_x = j_0 e^{i(\omega t - (1-i)/8)} \vec{u}_x$

$\Delta j = -\left(\frac{1-i}{8} \right)^2 j_0 e^{i(\omega t - (1-i)/8)}$

$\mu_0 \gamma \frac{\partial j}{\partial t} = \mu_0 \gamma i\omega j_0 e^{i(\omega t - (1-i)/8)}$

or $\delta^2 = \frac{2}{\mu_0 \gamma \omega} \rightarrow \left(\frac{1-i}{8} \right)^2 = \left(\frac{1-i}{\sqrt{2}} \right)^2 \mu_0 \gamma \omega$
 $= (e^{-i\pi/4})^2 \mu_0 \gamma \omega = -i \mu_0 \gamma \omega$

d'où le résultat.

$$2- a) \vec{j} = \gamma \vec{E} \rightarrow \vec{E} = \frac{\vec{j}}{\gamma}$$

$$\vec{E} = \frac{j_0}{\gamma} e^{-z/\delta} \exp(i(\omega t - \frac{z}{\delta})) \vec{u}_x$$

$$b) \text{rot } \vec{E} = - \frac{\partial \vec{B}}{\partial t}$$

$$\nabla \wedge \vec{E} = - i\omega \vec{B}$$

$$\begin{vmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{vmatrix} \wedge \begin{vmatrix} E_x(z,t) \\ 0 \\ 0 \end{vmatrix} = \begin{vmatrix} 0 \\ \frac{\partial E_x}{\partial z} \\ 0 \end{vmatrix} = \begin{vmatrix} 0 \\ -i\omega B_y \\ 0 \end{vmatrix} \rightarrow \vec{B} = \begin{vmatrix} B_x = 0 \\ B_y = \frac{-1}{i\omega} \frac{\partial E_x}{\partial z} \\ B_z = 0 \end{vmatrix}$$

$$\text{or } E_x = \frac{j_0}{\gamma} e^{i(\omega t - (1-i)\frac{z}{\delta})}$$

$$\frac{\partial E_x}{\partial z} = -i(1-i) \frac{j_0}{\gamma} e^{i(\omega t - (1-i)\frac{z}{\delta})}$$

$$\rightarrow B_y = \frac{-i(1-i)}{-i\omega \gamma} \frac{j_0}{\gamma} e^{i(\omega t - (1-i)\frac{z}{\delta})}$$

$$\rightarrow \vec{B} = \frac{j_0}{\gamma \delta \omega} (1-i) e^{-z/\delta} e^{i(\omega t - z/\delta)} \vec{u}_y$$

$$\text{or } \vec{B} = \frac{\sqrt{2} j_0}{\gamma \delta \omega} e^{-z/\delta} e^{i(\omega t - z/\delta - \pi/4)} \vec{u}_y$$

$$3- u = \epsilon_0 \frac{E^2}{2} + \frac{B^2}{2\mu_0} \quad (\text{on doit travailler en notation réelle})$$

$$= \frac{\epsilon_0}{2} \frac{j_0^2}{\gamma^2} e^{-2z/\delta} \cos^2(\omega t - \frac{z}{\delta}) + \frac{1}{2\mu_0} \frac{j_0^2}{\gamma^2 \delta^2 \omega^2} e^{-2z/\delta} \cos^2(\omega t - \frac{z}{\delta} - \pi/4)$$

$$u = \frac{j_0^2}{\gamma^2} e^{-2z/\delta} \left[\frac{\epsilon_0}{2} \cos^2(\omega t - \frac{z}{\delta}) + \frac{1}{\mu_0 \delta^2 \omega^2} \cos^2(\omega t - \frac{z}{\delta} - \pi/4) \right]$$

$$\langle u \rangle = \frac{j_0^2}{\gamma^2} e^{-2z/\delta} \left[\frac{\epsilon_0}{2} \times \frac{1}{2} + \frac{1}{\mu_0 \delta^2 \omega^2} \times \frac{1}{2} \right]$$

$$\text{or } \gamma^2 = \frac{2}{\mu_0 \delta \omega} \rightarrow \delta^2 \mu_0 = \frac{2}{\gamma \omega}$$

$$\rightarrow \langle u \rangle = \frac{j_0^2}{\gamma^2} e^{-2z/\delta} \left[\frac{\epsilon_0}{4} + \frac{\gamma}{4\omega} \right]$$

$$\frac{\langle \mu_e \rangle}{\langle \mu_m \rangle} = \frac{\epsilon_0 \omega}{\gamma}$$

$$= \frac{\epsilon_0 \omega}{\gamma} \ll 1$$

4- la puissance moyenne dissipée par effet Joule est

$$\langle P_j \rangle = \iiint_V \langle \vec{j} \cdot \vec{E} \rangle \cdot d\vec{\tau} = \int_0^\infty \langle \gamma E^2 \rangle a \cdot b \, dz$$

$$= \int_0^\infty \frac{j_0^2}{2\gamma} e^{-2z/\delta} a \cdot b \, dz$$

$$= \frac{j_0^2}{2\gamma} a \cdot b \left[-\frac{\delta}{2} e^{-2z/\delta} \right]_0^\infty$$

$$\langle P_j \rangle = \frac{j_0^2 a \cdot b \delta}{4\gamma}$$

5- la puissance moyenne rayonnée :

$$\langle P_r \rangle = \iint \langle \vec{R} \rangle \cdot d\vec{S} \quad \text{avec} \quad \langle \vec{R} \rangle = \langle \vec{E} \wedge \frac{\vec{B}}{\mu_0} \rangle = \frac{1}{2\mu_0} \text{Re}(\vec{E} \wedge \vec{B}^*)$$

on utilise cette formule car $\vec{E}$ et $\vec{B}$ ne sont pas en phase.

$$\langle \vec{R} \rangle = \frac{1}{2\mu_0} \text{Re} \left(\frac{j_0}{\gamma} e^{-z/\delta} e^{i(\omega t - z/\delta)} \vec{u}_n \wedge \frac{j_0}{\gamma \omega} (1+i) e^{-z/\delta} e^{-i(\omega t - z/\delta)} \vec{u}_y \right)$$

$$\langle \vec{R} \rangle = \frac{j_0^2}{2\mu_0 \gamma^2 \omega} e^{-2z/\delta} \vec{u}_z$$

$$\rightarrow \langle P_r \rangle = \frac{j_0^2 a \cdot b \delta}{2\mu_0 \gamma^2 \omega} e^{-2z/\delta} = \frac{j_0^2 e^{-2z/\delta} a b \cdot \delta}{4\gamma} = \langle P_j \rangle$$

à l'entrée du conducteur, c.à.d en $z=0$ on a $\langle P_r \rangle_{z=0} = \langle P_j \rangle$

6- toute l'énergie qui entre par la surface ab se trouve entièrement dissipée par effet Joule dans le conducteur.